Spatio-Temporal Dielectric Composites with Negative Values of ϵ and μ and Negative Energy Density

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This paper gives two ways of constructing spatio-temporal dielectric composites (dynamic materials) with negative effective permeability μ and/or permittivity ϵ. In particular, we obtain materials with negative energy density generated by the energy exchange with the environment. Such materials can be used toward an effective coupling of wave modes.

Keywords: activated and kinetic dynamic materials, spatio-temporal polycrystals, negative energy density

1. Introduction

Spatio-temporal material composites (dynamic materials) have been introduced and discussed in (Lurie 1997-2001, Blekhman and Lurie 2000) in both mechanical and electromagnetic contexts. The focus of this paper is on electromagnetic materials, specifically, isotropic dielectrics, and most of the discussion is related to binary composites assembled in space-time from two original constituents with material constants (ϵ_1, μ_1) (material 1), and (ϵ_2, μ_2) (material 2).

Two types of such composites were introduced in (Lurie 1998, 1999, Blekhman and Lurie 2000), and these were termed *activated* and *kinetic* dynamic materials. The difference between these types is best illustrated by an example.

Consider a transmission line. Its discrete version may be interpreted as an array of LC-cells connected in parallel (Fig. 1). Assume that each cell offers two possibilities: (L_1, C_1) and (L_2, C_2), turned on/off by a switch S. If the cells are densely distributed along the line, then, by due switching, the linear inductance L and capacitance C of the line may become, with any desired accuracy, almost arbitrary functions of the spatial coordinate z and time t. In particular, we may produce a periodic LC-laminate assembled from segments with properties (L_1, C_1) and (L_2, C_2), respectively (Fig. 2). In this figure, the pattern of such segments is shown moving along the z-axis at velocity V, and this motion creates the laminar structure in space-time. It is essential to note that this construction does not include any motion of the material itself; what is allowed to move, is the *property pattern* alone. This is a pure case of activation, and *activated* materials appear as a result of the homogenization procedure applied to this type of construction.

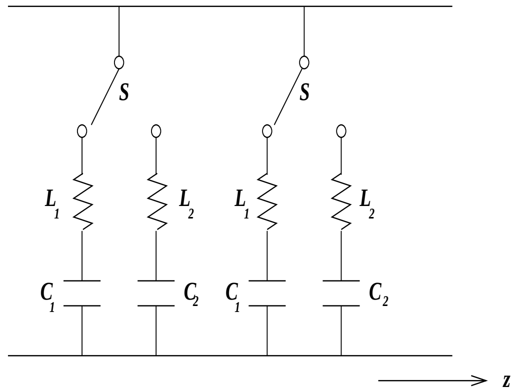

Figure 1. A discrete version of a transmission line

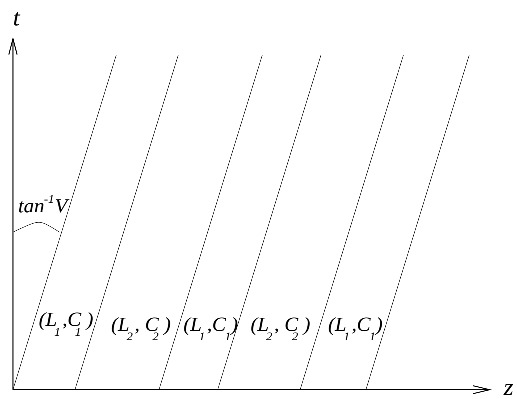

Figure 2. A moving (LC)-property pattern - an activated composite

Now consider a dielectric rod assembled from alternating sections of isotropic dielectrics with material constants (ϵ_1, μ_1) and (ϵ_2, μ_2), respectively (Fig. 3). As mentioned above, we term these dielectrics materials 1 and 2. Within each section, the material may be brought into its individual material motion along the z-axis at velocities v_1 (material 1) and v_2 (material 2). A discontinuous velocity pattern may be implemented either through the use of a special "caterpillar construction" introduced in (Lurie 2000) or, approximately, by a fast periodic longitudinal vibration of a dielectric continuum in the form of a standing wave. Contrary to the case of activation, the property pattern, i.e. the set of segments, now remains immovable in a laboratory frame; what is moving, is the *dielectric material itself* within the segments. This is a pure case of kinetization; a *kinetic* material appears after we apply homogenization to this type of construction.

In particular, when materials 1 and 2 are identical, the kinetic material turns out to be a spatio-temporal assemblage of fragments of the same original dielectric, where each fragment is brought into its own individual motion. For reasons explained below, this type of kinetic material was termed a spatio-temporal polycrystal.

In both activated and kinetic scenarios, homogenization introduces an averaged characterization of the composite material in terms of its effective constants. This

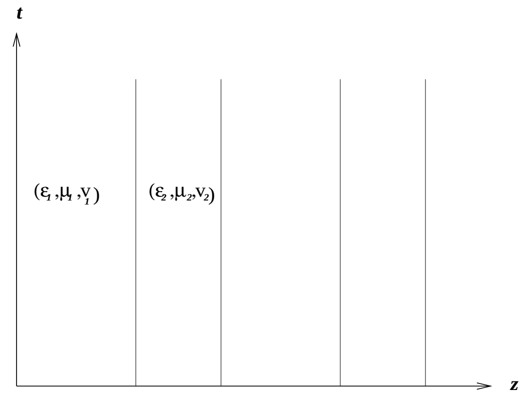

Figure 3. An immovable material pattern with moving original substances - a kinetic compsote

characterization is valid for disturbances whose wavelengths are long compared to the period of the material pattern.

The difference between activated and kinetic materials can be formalized in terms of the tensor s of their material properties (Lurie 1998). If an isotropic dielectric with properties (ϵ, μ) remains immovable in a Minkowskian laboratory frame $x_1 = x,\ \ x_2 = y,\ \ x_3 = z,\ \ x_4 = ict$, then its tensor s is specified by the formula

$$s = -\frac{1}{\mu c}(a_{12}a_{12} + a_{13}a_{13} + a_{23}a_{23}) - \epsilon c(a_{14}a_{14} + a_{24}a_{24} + a_{34}a_{34}), \qquad (1.1)$$

where $a_{mn} = (1/2)(e_m e_n - e_n e_m)$, $m, n = 1, \ldots, 4$ denote the eigentensors defined as skew-symmetric combinations of the unit vectors e_m of the Minkowskian x_m-axes. If the original material is moving with respect to a laboratory frame, then its material tensor is also given by equation (1.1), with symbols a_{mn} replaced by a'_{mn} formed by the vectors e'_m which are linked with e_m by a Lorentz transform. The material motion generates rotation of the eigenaxes of a'_{mn}, specifically, of the time axis x'_4, relative to the x_m-axes. As a result, this rotation produces the well-known Minkowski's material relations for a moving dielectric medium.

The tensor language formalizes the said difference between activated and kinetic dynamic composites. For activated composites, the original constituents differ *only in the eigenvalues* $1/\mu c, \epsilon c$ of their material tensors, whereas their eigentensors a_{mn} remain the same in the absence of a relative material motion. There may be, however, a common background "solid" material motion of the whole assemblage with respect to a laboratory frame, but this motion will never violate the identity of eigentensors. For kinetic composites, the original constituents differ also *in their eigentensors*; this difference becomes the *only* difference in the case of spatio-temporal polycrystals. The term polycrystal is appropriate because it reflects the difference in orientation of the eigenaxes of a_{mn} in the original fragments relative to the laboratory frame.

The following example gives an additional illustration of the contrast between activated and kinetic composites. Consider an activated laminate in one spatial dimension as represented in Figure 4. Once materials 1 and 2 are kept at rest within the layers, then the s-tensors differ *only in their eigenvalues.* This particular feature is characteristic of activation.

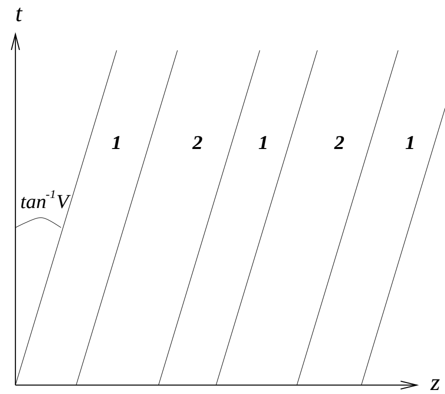

Figure 4. Material laminate in space-time.

Contrary to that, assume that the same material assemblage is brought as a whole into a *material motion* at the same velocity V along the z-axis; the property pattern will then also travel at the velocity V. Clearly, the microstructure in the (z,t)-plane will be given by Fig. 4 also. The difference is that, in the second case, the layers are occupied by moving materials, with their eigentensors accordingly modified: $a'_{mn} \neq a_{mn}$. The new tensors a'_{mn}, however, are the same for both materials because there is no relative material motion. Therefore, the second case appears to be a combination of *activation* produced by the *pattern moving* at the velocity V, and of *kinetization* produced by *material motion* occuring at the same velocity V *identical for both materials.* As a result, the effective material parameters $\mathcal{E}, M$ will be the same as they are for a static laminate with the same material mixture; more precisely, these parameters will not depend on V. In the first case, though, the effective properties are affected by the pattern's motion, i.e., they depend on V. In the second case this dependency is removed by the counter balancing effect of the material travelling within the layers at the same velocity V.

If we activate a pattern of materials, immovable in a laboratory frame, and apply homogenization, the effective tensor s of the composite will not be diagonal in a laboratory frame. To diagonalize it, we need to go to a proper (co-moving) coordinate frame such that the composite as a whole experiences actual material motion. The eigentensors a'_{mn} of such a composite will have their eigenaxes rotated relative to the original laboratory frame. Of course, this is true also with regard to composites obtained through kinetization; material motion affects eigentensors of any *given* material tensor. This observation should be borne in mind specifically with regard to constructing composites of higher rank.

2. An activated rank-one laminate in space-time

Consider a laminate (Fig. 4) assembled from two isotropic dielectrics characterized by different pairs of values of $\epsilon = \epsilon(z,t)$ and $\mu = \mu(z,t)$:

$$(\epsilon(z,t), \mu(z,t)) = \begin{cases} (\epsilon_1, \mu_1) & \text{material 1,} \\ (\epsilon_2, \mu_2) & \text{material 2.} \end{cases} \tag{2.1}$$

We may perceive this construction as the periodic array of segments of the z-axis carrying materials 1 and 2 and occupying, respectively, portions m_1 and m_2 of the period. The array (property pattern) is assumed moving along the z-axis at the velocity V which specifies the slope of the layers' interfaces in Fig. 4, whereas the dielectric materials that fill the segments remain at rest with respect to the laboratory frame.

The slope $V = dz/dt$ satisfies the inequality

$$(V^2 - a_1^2)(V^2 - a_2^2) \geq 0, \tag{2.2}$$

where $a_i = 1/\sqrt{\epsilon_i \mu_i}$ is the phase velocity of light in material i. This inequality is necessary to guarantee smoothness of the relevant solution (the non-appearance of shocks).

For one-dimensional wave propagation, the Maxwell's system

$$\text{curl } \mathbf{E} = -\mathbf{B}_t, \ \ \text{div } \mathbf{B} = 0, \ \ \text{curl } \mathbf{H} = \mathbf{D}_t, \quad \text{div } \mathbf{D} = 0,$$

is satisfied by the vectors

$$\mathbf{E} = u_t \mathbf{j}, \ \ \mathbf{B} = u_z \mathbf{i}, \ \ \mathbf{H} = v_t \mathbf{i}, \ \ \mathbf{D} = v_z \mathbf{j}, \tag{2.3}$$

representing a plane electromagnetic wave traveling in the z-direction. The material relations $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$ generate the system

$$\epsilon u_t = v_z, \ \ \frac{1}{\mu} u_z = v_t. \tag{2.4}$$

After homogenization, it is replaced by the system

$$\alpha c u_z + \beta u_t = V v_z + v_t, \tag{2.5}$$

$$V u_z + u_t = \theta(\alpha c v_z + \beta v_t), \tag{2.6}$$

with parameters α, β, θ defined as

$$\alpha = \frac{1}{c} \frac{\left\langle \frac{1}{\varepsilon\mu(V^2-a^2)} \right\rangle}{\left\langle \frac{1}{\varepsilon(V^2-a^2)} \right\rangle}, \quad \beta = V \frac{\left\langle \frac{1}{V^2-a^2} \right\rangle}{\left\langle \frac{1}{\varepsilon(V^2-a^2)} \right\rangle}, \quad \theta = \frac{\left\langle \frac{1}{\varepsilon(V^2-a^2)} \right\rangle}{\left\langle \frac{1}{\mu(V^2-a^2)} \right\rangle}, \tag{2.7}$$

where

$$\langle \cdot \rangle = m_1 (\cdot)_1 + m_2 (\cdot)_2,$$

and c is the speed of light in a vacuum.

In Eqs. (2.5), (2.6), we preserved the original symbols u, v to denote the weak limits of the relevant quantities, i.e. their values averaged over the cell of periodicity. An equivalent form of these equations is given by (Lurie 1998)

$$p u_z - q u_t = v_t, \quad q u_z + r u_t = v_z, \tag{2.8}$$

with parameters p, q, r defined as

$$p = \frac{V^2 - \theta \alpha^2 c^2}{\theta(\beta V - \alpha c)}, \quad q = -\frac{V - \theta \alpha c \beta}{\theta(\beta V - \alpha c)}, \quad r = -\frac{1 - \theta \beta^2}{\theta(\beta V - \alpha c)}. \tag{2.9}$$

Introduce the "primed" coordinate frame z', t' moving with velocity w with respect to the laboratory frame z, t. Coordinates z', t' are linked with z, t by the Lorentz formulae

$$z' = \gamma^{-1}(z - wt), \quad t' = \gamma^{-1}\left(t - \frac{w}{c^2}z\right), \quad \gamma = \sqrt{1 - w^2/c^2}. \tag{2.10}$$

If w is defined as the root of

$$\frac{q}{c^2}w^2 + \left(\frac{p}{c^2} - r\right)w + q = 0, \tag{2.11}$$

then the system (2.8) becomes diagonalized, i.e. reduced to equations

$$(p + qw)u_{z'} = v_{t'}, \quad \left(\frac{p}{c^2} + \frac{q}{w}\right)u_{t'} = v_{z'}, \tag{2.12}$$

specifying the effective parameters $\mathcal{E}, M$ (c.f. (2.4)) via the formulae for the eigenvalues of the material tensor of a composite:

$$\mathcal{E}c = \frac{p}{c} + \frac{qc}{w}, \quad \frac{1}{Mc} = \frac{p}{c} + \frac{qw}{c}. \tag{2.13}$$

Applying direct inspection and referring to (2.11), we obtain the following expression for the second invariant of a material tensor:

$$\frac{\mathcal{E}}{M} = \left(\frac{p}{c} + \frac{qc}{w}\right)\left(\frac{p}{c} + \frac{qw}{c}\right) = pr + q^2,$$

so by Eqs. (2.9),

$$\frac{\mathcal{E}}{M} = pr + q^2 = 1/\theta. \tag{2.14}$$

We will also need the formula for the first invariant $\mathcal{E}c + 1/Mc$ of the effective tensors of material parameters. This formula follows from (2.13) and (2.11):

$$\mathcal{E}c + 1/Mc = \frac{2p}{c} + \frac{qc}{w} + \frac{qw}{c} = \frac{p}{c} + rc. \tag{2.15}$$

Given Eqs. (2.9), we rewrite (2.15) in the form

$$\mathcal{E}c + 1/Mc = \frac{1}{\beta(V/c) - \alpha}\left[\left(\frac{V^2}{c^2} - 1\right)\frac{1}{\theta} - \left(\alpha^2 - \beta^2\right)\right]. \tag{2.16}$$

We now summarize restrictions that should be observed when operating with these formulae. We first assume that parameters $\epsilon_1, \ldots, \mu_2$ are all positive. Without a loss of generality, set $a_2^2 > a_1^2$, that is $\epsilon_1\mu_1 > \epsilon_2\mu_2$. This ordering holds true if we are either in the *regular* case where $\epsilon_1 > \epsilon_2, \mu_1 > \mu_2$ or in the *irregular* case where $\epsilon_1 > \epsilon_2, \mu_1 < \mu_2$ (or $\epsilon_1 < \epsilon_2, \mu_1 > \mu_2$). These possibilities will affect the admissible values of some characteristic parameters listed below.

Define the symbol $\bar{\epsilon}$ as

$$\bar{\epsilon} = m_1\epsilon_2 + m_2\epsilon_1,$$

and similarly introduce the symbols $\bar{\mu}, (\overline{1/\epsilon}), (\overline{1/\mu})$. It is easily checked that, for positive ϵ, μ,

$$\begin{aligned} &\bar{\epsilon}(\overline{1/\epsilon}) \geq 1, \\ &\bar{\mu}(\overline{1/\mu}) \geq 1, \\ &a_1^2 \leq (1/\bar{\epsilon})(\overline{1/\mu}) \leq a_2^2, \\ &a_1^2 \leq (1/\bar{\mu})(\overline{1/\epsilon}) \leq a_2^2. \end{aligned} \tag{2.17}$$

Also,

$$\begin{aligned} (1/\bar{\epsilon})(1/\bar{\mu}) &\leq a_2^2, \\ (\overline{1/\epsilon})(\overline{1/\mu}) &\geq a_1^2, \end{aligned} \tag{2.18}$$

for both the regular and irregular cases; and, for the regular case,

$$\begin{aligned} (1/\bar{\epsilon})(1/\bar{\mu}) &\geq a_1^2, \\ (\overline{1/\epsilon})(\overline{1/\mu}) &\leq a_2^2. \end{aligned} \tag{2.19}$$

As for the irregular case, there exists (Lurie 1997) a range of m_1 for which

$$(1/\bar{\epsilon})(1/\bar{\mu}) \leq a_1^2, \tag{2.20}$$

and a range of m_1 for which

$$(\overline{1/\epsilon})(\overline{1/\mu}) \geq a_2^2. \tag{2.21}$$

For example, if $\epsilon_1 = \mu_1 = 1,\ \ \epsilon_2 = 9,\ \ \mu_2 = 0.1$, then we have the irregular case, and (2.20) holds if $m_1 \leq 71/72$, and (2.21) holds if $m_1 \geq 1/72$. Both inequalities are satisfied when $1/72 \leq m_1 \leq 71/72$.

Aside from the restrictions in (2.2),

$$V^2 \leq a_1^2 \ \text{ or } \ V^2 \geq a_2^2, \tag{2.22}$$

specifying the slow and fast ranges for V^2, we note a universal inequality $V^2 \leq c^2$. The roots, w, of equation (2.11) should be real. Since their product equals c^2, one of those roots has absolute value less than or equal to c; this particular root participates in the Lorentz transform (2.10). We thus demand that the discriminant of (2.11) be non-negative:

$$\left(\frac{p}{c^2} - r\right)^2 - 4\frac{q^2}{c^2} \geq 0,$$

or, equivalently,

$$\left(\frac{p}{c} + rc\right)^2 - 4(pr + q^2) \geq 0.$$

Given (2.14) and (2.15), this is

$$\left(\mathcal{E}c + \frac{1}{Mc}\right)^2 - 4\frac{\mathcal{E}}{M} \geq 0. \tag{2.23}$$

When the roots of (2.11) are real, so are $\mathcal{E}c, 1/Mc$.

We now consider the explicit expressions for the first and the second invariants $I_1 = \mathcal{E}c + 1/Mc$, $I_2 = \mathcal{E}/M$ of the effective material tensor s (Lurie 1998). Using (2.7), (2.14), and (2.15), we find after some calculation,

$$I_1 = \mathcal{E}c + \frac{1}{Mc} = \frac{\bar{\epsilon}\bar{\mu} + \frac{c^2}{a_1^2 a_1^2}}{\bar{\epsilon}\mu_1\mu_2 c\left[V^2 - \frac{1}{\bar{\epsilon}}\left(\overline{\frac{1}{\mu}}\right)\right]}(V^2 - k), \tag{2.24}$$

$$k = \frac{c^2\left(\overline{\frac{1}{\epsilon}}\right)\left(\overline{\frac{1}{\mu}}\right) + a_1^2 a_2^2}{c^2 + a_1^2 a_2^2 \bar{\epsilon}\bar{\mu}}, \tag{2.25}$$

$$I_2 = \frac{\mathcal{E}}{M} = \frac{\langle\frac{1}{\mu}\rangle}{\langle\frac{1}{\epsilon}\rangle}\,\frac{V^2 - \frac{1}{\bar{\mu}}\left(\overline{\frac{1}{\epsilon}}\right)}{V^2 - \frac{1}{\bar{\epsilon}}\left(\overline{\frac{1}{\mu}}\right)}, \tag{2.26}$$

Remark 1 It is easy to see that $k < c^2$. Assuming the contrary, gives the inequality $f(c^2) < 0$ where $f(\lambda)$ is defined as

$$f(\lambda) \equiv \lambda^2 + \lambda\left[a_1^2 a_2^2 \bar{\epsilon}\bar{\mu} - \left(\overline{\frac{1}{\epsilon}}\right)\left(\overline{\frac{1}{\mu}}\right)\right] - a_1^2 a_2^2.$$

Clearly, the equation $f(\lambda) = 0$ has real roots of opposite signs, the product of these roots being $-a_1^2 a_2^2$. We check that $f(a_1^2) = 2m_1 a_1^2(a_1^2 - a_2^2) < 0$, and $f(a_2^2) = 2m_2 a_2^2(a_2^2 - a_1^2) > 0$; this means that a_2^2 exceeds the positive root of $f(\lambda) = 0$, and $f(\lambda) > 0$ for $\lambda \geq a_2^2$. Since $c^2 > a_2^2$, we conclude that $f(c^2) > 0$, and hence our assumption is false.

Using (2.24)–(2.26), inequality (2.23) takes on the form:

$$\left(\bar{\epsilon}\bar{\mu} + \frac{c^2}{a_1^2 a_2^2}\right)^2 (V^2 - k)^2 \geq 4\frac{c^2}{a_1^2 a_2^2}\bar{\epsilon}\bar{\mu}\left[V^2 - \frac{1}{\bar{\mu}}\left(\overline{\frac{1}{\epsilon}}\right)\right]\left[V^2 - \frac{1}{\bar{\epsilon}}\left(\overline{\frac{1}{\mu}}\right)\right]. \tag{2.27}$$

Define the parameter σ as

$$\sigma = \frac{c^2}{a_1^2 a_2^2 \bar{\epsilon}\bar{\mu}}, \tag{2.28}$$

and rewrite (2.25) as

$$k = \frac{\sigma\bar{\epsilon}\bar{\mu}\left(\overline{\frac{1}{\epsilon}}\right)\left(\overline{\frac{1}{\mu}}\right) + 1}{\sigma + 1}\,\frac{1}{\bar{\epsilon}\bar{\mu}}. \tag{2.29}$$

Inequality (2.27) is now rewritten as

$$(1+\sigma)^2(V^2 - k)^2 \geq 4\sigma\left[V^2 - \frac{1}{\bar{\mu}}\left(\overline{\frac{1}{\epsilon}}\right)\right]\left[V^2 - \frac{1}{\bar{\epsilon}}\left(\overline{\frac{1}{\mu}}\right)\right]. \tag{2.30}$$

We now look for the possibility for parameters $\mathcal{E}c, 1/Mc$ to become negative. Eqs. (2.24)–(2.30) are valid for arbitrary ϵ, μ; however, we first assume that all $\epsilon, \mu > 0$. The product (2.26) is then non-negative given (2.22) and (2.17). As to the sum (2.24), it may be negative if either (i) $V^2 < (1/\bar{\epsilon})(\overline{1/\mu})$ and $V^2 > k$, or (ii) $V^2 > (1/\bar{\epsilon})(\overline{1/\mu})$ and $V^2 < k$. The second possibility can be made consistent with (2.30), as shown by the following argument. Referring to (2.22) and to (2.17), we conclude that V^2 should be taken greater than a_2^2. This may come to agreement with $V^2 < k$ since k may exceed a_2^2 if the value of σ is sufficiently large. In fact, if $\sigma \to \infty$, then k monotonically increases approaching the value $\overline{(1/\epsilon)}\,\overline{(1/\mu)}$ *which may exceed* a_2^2 *for the irregular case* (see (2.21)). Considering this case and choosing V^2 within the interval (a_2^2, k), we observe that, for sufficiently large values of σ, the LHS of (2.30) prevails.

This conclusion was based on the assumption that ϵ, μ are positive for the original materials. When these constants are negative, then the same argument shows that the effective parameters $\mathcal{E}, M$ may become positive. This follows directly from (2.25).

Directly computing the numerical solution to wave propagation through the fast range ($V^2 > a_2^2$) dynamic materials has proven to be an interesting problem. A more standard conservative finite difference approach analogous to the one taken in (Weekes 2001a) for the slow range ($V^2 < a_1^2$) and for static laminates yields an unstable scheme. Numerical results are degraded since accuracy is quickly lost due to the growth of short waves which enter into the computation as truncation and round-off error. In (Weekes 2001b), an approach is taken that successfully circumvents the appearance of these instabilities in the case of temporal laminates.

For the fast range laminates, we make the following change of coordinates: $\tau = t - \frac{z}{V}, \zeta = z$ yielding the PDE system (c.f. (2.4))

$$\epsilon u_\tau + \frac{1}{V} v_\tau = v_\zeta,$$
$$\mu v_\tau + \frac{1}{V} u_\tau = u_\zeta.$$

In terms of characteristic variables, this is the convection system

$$(u + v/\nu)_\tau - \frac{aV}{V+a}(u + v/\nu)_\zeta = 0 \tag{2.31}$$

$$(u - v/\nu)_\tau + \frac{aV}{V-a}(u - v/\nu)_\zeta = 0, \tag{2.32}$$

where $\nu = \sqrt{\epsilon/\mu}$ is the material impedance, and a is the phase velocity of the material. In ζ, τ coordinates, the fast range dynamic material is as a temporal material where the property pattern depends on τ alone and has period ε. When a wave is incident on the pattern interfaces, $\tau = n\varepsilon$ or $\tau = (n + m_1)\varepsilon$ for n an integer, two new waves arise which both move into the new material. These waves are of the same wave number as the incident wave when looked upon in the new coordinate system. However, short wave modes unavoidably introduced into the computation will grow and destroy the fidelity of the results. We perform a spectral decomposition of the initial data, and at very regular intervals in the course of the numerical computation, we filter out those wave modes that lie without the range initially present. This spectral approach has proved successful and we illustrate some of the results below.

In Figures 5 and 6, we show contour plots of the results of propagating an initial Gaussian pulse,

$$u(z,0) = e^{-z^2}, \quad v(z,0) = 0,$$

through a fast range dynamic laminate with material parameters

$$\epsilon_1 = \mu_1 = 1, \;\; \epsilon_2 = 9, \;\; \mu_2 = 0.1, \quad m_1 = 0.5.$$

The figures show the results in z, t coordinates when $V = 4$ and $V = 1.3$. The horizontal axis gives the z-values, while time is on the vertical axis. Fig. 7 represents the plot of $1/M$ versus $\mathcal{E}$, with V variable along the curve. A model value for c is taken equal to $10a_2$. The interval (1.1841, 1.5850) of V corresponds to negative values of both effective parameters.

While $V = 4$ lies in the range of pattern velocities that give a homogenized material with positive effective coefficients, the material resulting from $V = 1.3$ has negative effective coefficients. The coordinated wave motion that arises in the latter case is clearly visible in the contour plot. One may check that the group velocities predicted by the computations match those predicted by the theory. For $V = 1.3$, the theory predicts the velocities to be 1.09324264 and 2.71473084, and for $V = 4$, the velocities are 1.40012760 and -2.63617910 for the combination of parameters used in this example.

We note that these are the results that come from the direct, detailed computation of the unhomogenized equations, and not from computing solutions to the effective equations.

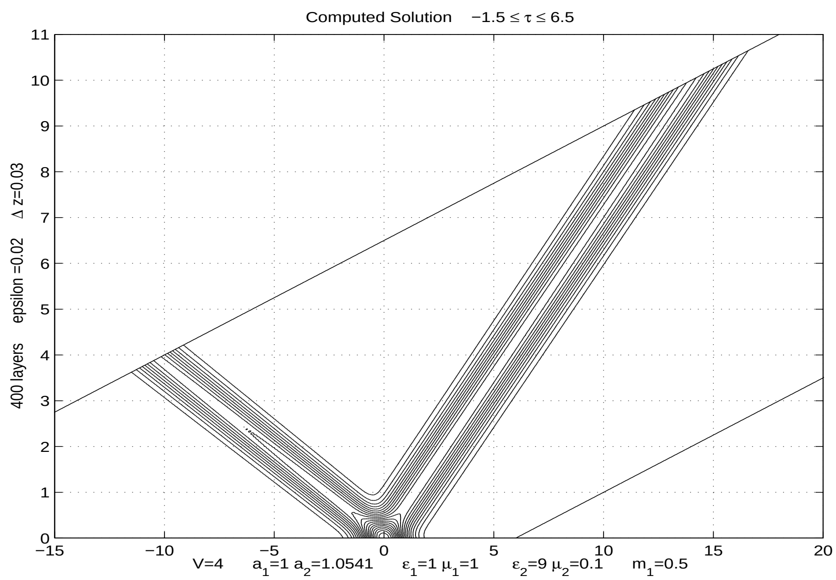

Figure 5. Wave propagation through a fast range laminate where $V = 4$ yields a homogenized material with positive effective coefficients.

3. A spatio-temporal laminar polycrystal with material motion maintained along the interfaces

Consider plane electromagnetic waves propagating through a periodic array composed of two types of layers perpendicular to the z-axis which are occupied by an isotropic dielectric with properties ϵ, μ. The dielectric is moving along the x-axis at the speed v_1 in layers of the first type, and at the speed v_2 in layers of the second type. Two types of layers are represented in a microstructure with the volume

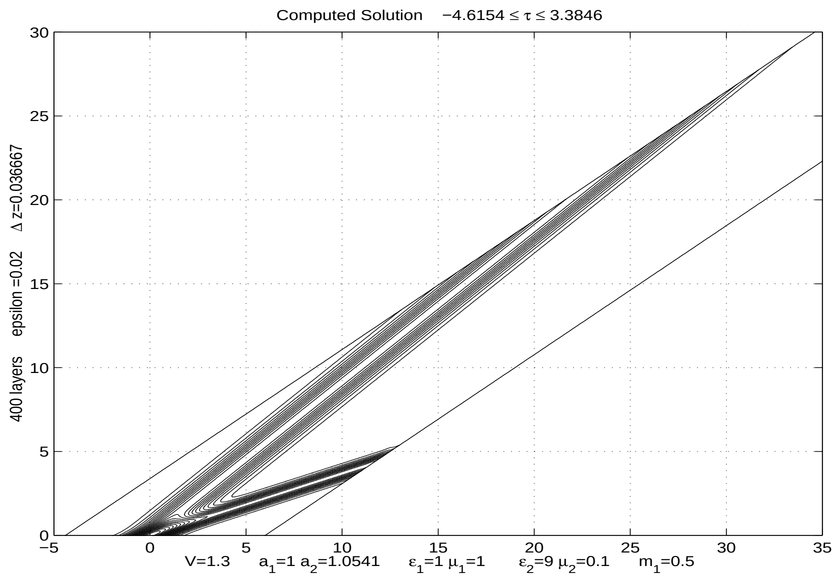

Figure 6. Wave propagation through a fast range laminate where $V = 1.3$ yields a homogenized material with negative effective coefficients.

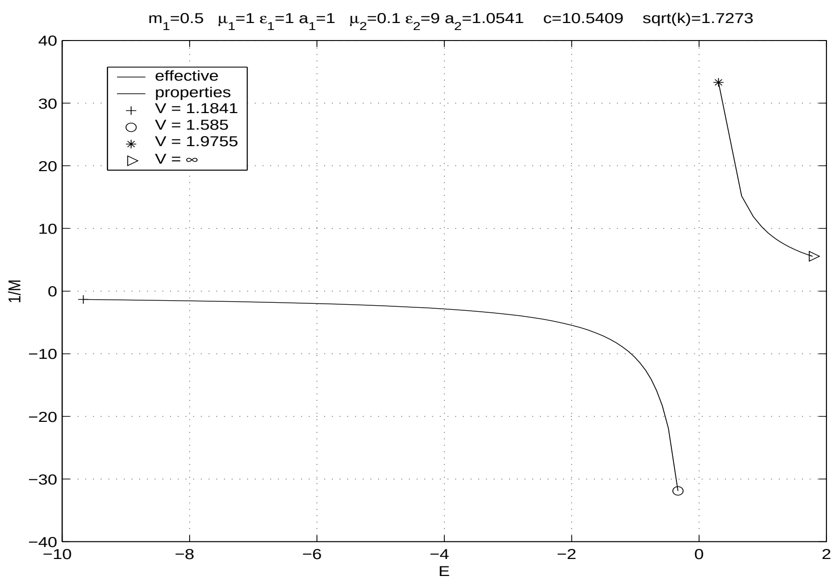

Figure 7. Effective properties in an electromagnetic material.

fractions m_1 and m_2, respectively. The waves have the plane (x, z) as their plane of incidence; the electric vector $\mathbf{E}$ is parallel to the y-axis. The electromagnetic field is characterized as

$$\mathbf{E} = E_2\mathbf{j}, \quad \mathbf{B} = B_1\mathbf{i} + B_3\mathbf{k}, \quad \mathbf{H} = H_1\mathbf{i} + H_3\mathbf{k}, \quad \mathbf{D} = D_2\mathbf{j}, \tag{3.1}$$

with the corresponding tensors, F and f, (Lurie 1998) given by

$$\begin{aligned} F &= cB_3a_{12} + cB_1a_{23} - iE_2a_{24}, \\ f &= H_3a_{12} + H_1a_{23} - icD_2a_{24}. \end{aligned}$$

Here,

$$\begin{aligned} a_{12} &= (1/\sqrt{2})(e_1e_2 - e_2e_1), \\ a_{23} &= (1/\sqrt{2})(e_2e_3 - e_3e_2), \\ a_{24} &= (1/\sqrt{2})(e_2e_4 - e_4e_2), \end{aligned} \tag{3.2}$$

and $e_1, \ldots, e_4$ denote the unit vectors of Minkowskian coordinate frame, $x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict$. Note that

$$a_{ik} : a^T_{\ell m} = \begin{cases} 1, & i = \ell, k = m, \\ 0 & \text{otherwise}, \end{cases}$$

i.e. the tensors a_{ik} constitute an orthonormal set.

The material tensor s entering the material relation

$$f = s : F, \tag{3.3}$$

is given within the layers by the expression

$$s = -\frac{1}{\mu c}a'_{12}a'_{12} - \frac{1}{\mu c}a'_{23}a'_{23} - \epsilon c a'_{24}a'_{24}, \tag{3.4}$$

where the unit tensors $a'_{12}, a'_{23}, a'_{24}$ are defined by the formulae (3.2) with the unit vectors $e_1, \ldots e_4$ replaced by

$$e'_1 = e_1\cosh\phi + ie_4\sinh\phi,\ e'_2 = e_2,\ e'_3 = e_3,\ e'_4 = -ie_1\sinh\phi + e_4\cosh\phi,$$

and the angle ϕ specified by

$$\tanh\phi = \begin{cases} \tanh\phi_1 = v_1/c & \text{for layers of the 1st type}, \\ \tanh\phi_2 = v_2/c & \text{for layers of the 2nd type}. \end{cases}$$

An equivalent expression for s takes on the form

$$s = -Aa_{12}a_{12} + iG(a_{12}a_{24} + a_{24}a_{12}) + Ca_{24}a_{24} - \frac{1}{\mu c}a_{23}a_{23},$$

where

$$\begin{aligned} A &= \frac{1}{\mu c}\cosh^2\phi - \epsilon c\sinh^2\phi, \\ G &= \left(\frac{1}{\mu c} - \epsilon c\right)\sinh\phi\cosh\phi, \\ C &= \frac{1}{\mu c}\sinh^2\phi - \epsilon c\cosh^2\phi, \end{aligned}$$

and $\phi = \phi_1$ or ϕ_2 in the relevant layers.

The material relation (3.3) is reduced within each layer to the system

$$\begin{aligned} AcB_3 &- GE_2 = H_3, \\ CE_2 &- GcB_3 = -cD_2, \\ \frac{1}{\mu}B_1 &= H_1. \end{aligned} \tag{3.5}$$

The Maxwell's equations take the form:

$$E_{2z} = B_{1t},\ E_{2x} = -B_{3t},\ B_{1x} + B_{3z} = 0,\ H_{1z} - H_{3x} = D_{2t}. \tag{3.6}$$

The first three equations will be satisfied by introducing potential u through the formulae

$$E_2 = u_t, \quad B_1 = u_z, \quad B_3 = -u_x. \tag{3.7}$$

These expressions should be applied after we eliminate H_1, H_3, D_2 from the fourth equation in (3.6) with the aid of (3.5).

We now subject the system (3.5) and the fourth equation in (3.6) to homogenization; to this end, observe that the components E_2, B_3, and H_1 are continuous across the layers' interfaces, and therefore remain unaffected by homogenization. The homogenized equations take the form:

$$\begin{aligned} \langle A\rangle cB_3 - \langle G\rangle E_2 &= \langle H_3\rangle, \\ \langle C\rangle E_2 - \langle G\rangle cB_3 &= -c\langle D_2\rangle, \\ \langle B_1\rangle &= \langle \mu\rangle H_1, \\ H_{1z} - \langle H_3\rangle_x &= \langle D_2\rangle_t. \end{aligned} \tag{3.8}$$

Applying the same procedure to (3.7), we get

$$E_2 = \langle u\rangle_t, \quad \langle B_1\rangle = \langle u\rangle_z, \quad B_3 = -\langle u\rangle_x. \tag{3.9}$$

Equations (3.8) and (3.9) are reduced to a second order equation for $\langle u\rangle$; preserving the symbol u for this quantity, we reproduce the result in the form:

$$\left(\frac{1}{\langle\mu\rangle}u_z\right)_z + c(\langle A\rangle u_x)_x + [(\langle G\rangle u_t)_x + (\langle G\rangle u_x)_t] + \frac{1}{c}(\langle C\rangle u_t)_t = 0.$$

Assuming that the averaged values $\langle\mu\rangle, \ldots, \langle C\rangle$ do not depend upon the slow variables x, z, t, we rewrite this equation as

$$\frac{1}{\langle\mu\rangle}u_{zz} + c\langle A\rangle u_{xx} + 2\langle G\rangle u_{tx} + \frac{1}{c}\langle C\rangle u_{tt} = 0.$$

By a standard argument, we associate with this equation an effective material tensor (cf. (3.4))

$$s^0 = -\frac{1}{Mc}a^0_{12}a^0_{12} - \frac{1}{\langle\mu\rangle c}a_{23}a_{23} - \mathcal{E}ca^0_{24}a^0_{24},$$

with some orthonormal set $a^0_{12}, a_{23}, a^0_{24}$ of eigentensors, and eigenvalues $1/Mc$, $1/\langle\mu\rangle c$, $\mathcal{E}c$; here, $\mathcal{E}$ and M are defined by the relations

$$\mathcal{E}c + \frac{1}{Mc} = \langle A - C\rangle = \epsilon c + \frac{1}{\mu c}, \tag{3.10}$$

$$\mathcal{E}/M = \langle G\rangle^2 - \langle A\rangle\langle C\rangle. \tag{3.11}$$

Equation (3.10) shows that the sum $\epsilon c + 1/\mu c$ is preserved through homogenization; as to the product $\mathcal{E}/M$, this one may be made negative even with both ϵ and μ assumed positive. For example, take $m_1 = m_2 = 1/2, \phi_1 = -\phi_2$; then $\langle G\rangle = 0$, and

$$\langle A\rangle = A_1 = (1/\mu c)\cosh^2\phi_1 - \epsilon c\sinh^2\phi_1,$$
$$\langle C\rangle = C_1 = (1/\mu c)\sinh^2\phi_1 - \epsilon c\cosh^2\phi_1.$$

Because $\epsilon\mu c^2 > 1$, we have $C_1 < 0$, and A_1 also becomes negative when ϕ_1 is sufficiently large; in view of equation (3.11), this is a desired result. The effective parameters $\mathcal{E}, M$ are of opposite signs even though ϵ and μ are both assumed positive.

4. The energy considerations

While the effective parameters of the spatio-temporal composite may become negative, the averaged value of the electromagnetic energy density measured in a laboratory frame remains positive. For the case of one-dimensional wave propagation through a rank one laminate as considered in section 2, the average electric and magnetic energy densities are respectively defined by the formulae

$$\langle w_e\rangle = \frac{1}{2}\langle \mathbf{ED}\rangle = \frac{1}{2}\langle \epsilon u_t^2\rangle, \tag{4.1}$$

$$\langle w_m\rangle = \frac{1}{2}\langle \mathbf{BH}\rangle = \frac{1}{2}\left\langle \frac{1}{\mu}u_z^2\right\rangle. \tag{4.2}$$

Because of the continuity of u and v across the layers' interface, the derivatives

$$\begin{aligned} u_\tau &= u_t + Vu_z, \\ v_\tau &= v_t + Vv_z = \epsilon V u_t + \frac{1}{\mu}u_z \end{aligned} \tag{4.3}$$

are also continuous. We use (4.3) to express u_t, u_z as functions of ϵ, μ, V, and the continuous derivatives, u_τ, v_τ:

$$\begin{aligned} u_t &= -\frac{a^2 u_\tau}{V^2 - a^2} + \frac{V v_\tau}{\epsilon(V^2 - a^2)}, \\ u_z &= \frac{V u_\tau}{V^2 - a^2} - \frac{v_\tau}{\epsilon(V^2 - a^2)}. \end{aligned}$$

The value of $\langle w_e\rangle$ is thus calculated as

$$\begin{aligned} \langle w_e\rangle &= \frac{1}{2}\langle \epsilon u_t^2\rangle = \frac{1}{2}\left\langle \epsilon\left(\frac{a^2}{V^2 - a^2}\right)^2\right\rangle u_\tau^2 - \left\langle \frac{a^2}{(V^2 - a^2)^2}\right\rangle V u_\tau v_\tau \\ &+ \frac{1}{2}\left\langle \frac{1}{\epsilon(V^2 - a^2)^2}\right\rangle V^2 v_\tau^2; \end{aligned} \tag{4.4}$$

the derivatives u_τ, v_τ remain unaffected by averaging, and are identical with their averaged values. The latter are linked with the averaged values u_t, u_z, v_t, v_z through the formulae (see (2.8) and (2.9))

$$\begin{aligned} u_\tau &= u_t + V u_z, \\ v_\tau &= v_t + V v_z = (p + qV)u_z - (q - rV)u_t = \alpha c u_z + \beta u_t. \end{aligned} \tag{4.5}$$

Note that these formulae relate the *averaged* values of $u_z, \ldots, v_t$, and are therefore different from those in (4.3) which relate pointwise values and hold along the layers' interfaces.

By eliminating u_τ, v_τ from (4.4) with the aid of (4.5), we arrive at the following expression for $\langle w_e \rangle$:

$$\begin{aligned} \langle w_e \rangle &= \frac{1}{2}\left\langle \epsilon \frac{\left(\frac{V}{\epsilon}\beta - a^2\right)^2}{(V^2 - a^2)^2} \right\rangle u_t^2 + \left\langle \frac{\epsilon}{(V^2 - a^2)^2}\left(\frac{V}{\epsilon}\beta - a^2\right)\left(\frac{c\alpha}{\epsilon} - a^2\right) \right\rangle V u_t u_z \\ &+ \frac{1}{2}\left\langle \epsilon \frac{\left(\frac{c\alpha}{\epsilon} - a^2\right)^2}{(V^2 - a^2)^2} \right\rangle V^2 u_z^2. \end{aligned} \tag{4.6}$$

By a similar argument, we calculate $\langle w_m \rangle$ as

$$\begin{aligned} \langle w_m \rangle &= \frac{1}{2}\left\langle \frac{1}{\mu} \frac{\left(V - \frac{\beta}{\epsilon}\right)^2}{(V^2 - a^2)^2} \right\rangle u_t^2 + \left\langle \frac{1}{\mu} \frac{\left(V - \frac{\beta}{\epsilon}\right)\left(V^2 - \frac{c\alpha}{\epsilon}\right)}{(V^2 - a^2)^2} \right\rangle u_t u_z \\ &+ \frac{1}{2}\left\langle \frac{1}{\mu} \frac{\left(V^2 - \frac{c\alpha}{\epsilon}\right)^2}{(V^2 - a^2)^2} \right\rangle u_z^2. \end{aligned} \tag{4.7}$$

It is easily checked that

$$\langle w_e \rangle - \langle w_m \rangle = \langle w_e - w_m \rangle = \frac{1}{2} r u_t^2 + q u_t u_z - \frac{1}{2} p u_z^2, \tag{4.8}$$

the latter expression being the averaged action density. The action density is known to be quasiaffine (Lurie 1998), and in this capacity it ultimately specifies the effective material tensor. The averaged action density (4.8) serves as the integrand for the functional

$$\iint \langle w_e - w_m \rangle dz dt,$$

generating (2.8) as the Euler equations.

The energy-momentum tensor associated with this density has the *tt*-component given by

$$T^{tt} = u_t \frac{\partial \langle w_e - w_m \rangle}{\partial u_t} - \langle w_e - w_m \rangle = \frac{1}{2} r u_t^2 + \frac{1}{2} p u_z^2.$$

This is interpreted as the overall averaged energy density, measured in a laboratory frame, of the system defined as the union of the wave, the medium, and the *external agent* affecting the property pattern. For $V \neq 0$ or $V \neq \infty$, this averaged energy density is *not equal* to the sum $\langle w_e + w_m \rangle$, which represents, also in a laboratory frame, the energy of the wave and medium combined, with no contribution of the external agent. The reason for this phenomenon is, of course, due to the homogenization procedure that preserves the averaged value of original action density $(1/2)(u_z v_t - u_t v_z)$ as the same expression calculated this time for the *averaged* values u_z, u_t, v_z, v_t (the quasiaffinity property). For the energy density, however, this property does not hold true.

The analysis of section 2 has shown how to create dynamic materials with negative effective properties $\mathcal{E}, M$. The energy density

$$T^{t't'} = \frac{1}{2}\mathcal{E} u_{t'}^2 + \frac{1}{2M} u_{z'}^2$$

of such a material, measured in its proper coordinate frame (z', t'), is negative if $\mathcal{E}, M < 0$. The energy flux density

$$T^{t'z'} = (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{k} = -\frac{1}{M} u_{t'} u_{z'}$$

also changes its sign to opposite, and the group velocity $T^{t'z'}/T^{t't'}$ remains identical with the phase velocity $1/\sqrt{\mathcal{E}M}$. The negative wave energy may appear when a uniform dielectric with positive material constants ϵ, μ is moving with velocity V relative to an immovable observer. In the absence of motion ($V = 0$), the observer registers two waves of positive energy travelling with phase velocities $\pm a$ in opposite directions. In the presence of motion ($V > 0$), these waves are viewed by the observer, in his own coordinate frame, as a "fast" wave travelling with the velocity

$$\frac{V + a}{1 + Va/c^2},$$

and a "slow" wave with velocity

$$\frac{V - a}{1 - Va/c^2}.$$

Of these waves, only the slow wave possesses the negative energy density when $V > a$ (Chu 1951, Sturrock 1960, Pierce 1974); the energy of the fast wave remains positive. The group velocity of either wave coincides with its phase velocity.

In our situation, an observer that is immovable in a proper coordinate frame (z', t'), registers waves moving with the phase velocities $\pm 1/\sqrt{\mathcal{E}M}$ in opposite directions. Both waves now carry negative energy, and either of them can be coupled with the matching wave propagating along the adjacent transmission line (Louisell 1960, Pierce 1974), as it occurs in a travelling wave tube (TWT). A novel feature is that we now have *two* waves that may pump energy into transmission lines, whereas, in a standard TWT situation, we have only one such wave.

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