Low Frequency Longitudinal Vibrations of an Elastic Bar Made of a Dynamic Material and Excited at One End

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The dynamic materials, particularly, the spatio-temporal composites, have been investigated in a number of recent publications [1-5]. In the present paper, we examine a dynamic performance of an elastic bar with material parameters specified as an activated (moving at uniform velocity V) periodic pattern of segments occupied by ordinary materials 1 and 2, with density ρ and stiffness k specified as (ρ_1, k_1) and (ρ_2, k_2), respectively. We particularly consider the low frequency wave propagation arising when the period d of the material pattern is much less than the wavelength λ of a dynamic disturbance. The bar is excited at its left end by a signal $g(t)$.

1. STATEMENT OF THE PROBLEM

The longitudinal wave propagation along the elastic bar is governed by the equation

$$(\rho z^1_t)_t - (k z^1_x)_x = 0, \tag{1}$$

where z denotes the displacement of a material element, and symbols ρ, k are used for the material density and stiffness. An equivalent first order system of equations reads

$$\rho z^1_t = z^2_x, \quad k z^1_x = z^2_t. \tag{2}$$

We assume that the material parameters ρ, k address the following properties:

(i) they are both space and time dependent;

(ii) at each point (x, t) the pair (ρ, k) may take either the values (ρ_1, k_1) or the values (ρ_2, k_2); we specify these characterizations as "material 1" and "material 2," respectively;

(iii) an elastic bar is activited [1,2], i.e. materials 1 and 2 are applied within alternating layers in the (x, t)-plane, the slope $dx/dt = V$ of these layers being so chosen as to ensure observance of both kinematic and dynamic compatibility conditions across the interfaces separating the layers. These conditions demand that both z^1 and z^2 be continuous across any such interface; they will be satisfied if we postulate the following relationship between V and the phase velocities $a_i = \sqrt{k_i/\rho_i}$, $i = 1, 2$ $(a_2 > a_1)$ in materials 1 and 2 [1]:

$$(V^2 - a_1^2)(V^2 - a_2^2) \geq 0. \tag{3}$$

In this paper, we resort to the Laplace transform to examine solutions to the original system (2) with coefficients ρ, k defined as the periodic functions of the fast variable ξ/d, $\xi = x - Vt$. We shall be particularly interested in the behavior of such solutions near the left end $x = 0$ of a semi-infinite bar: $0 \leq x < \infty$; at this end, we apply the excitation condition

$$z^1(0, t) = g(t), \quad t > 0. \tag{4}$$

The dynamics of the wave propagation will be examined under zero initial conditions:

$$z^1(x, 0) = 0, \quad z^2(x, 0) = 0, \quad 0 \leq x < \infty.$$

2. AN ACTIVATED ELASTIC BAR: GENERAL FORMULAE

By introducing the new variables

$$\xi = x - Vt, \quad \tau = t, \tag{5}$$

we reduce the system (2) to the form

$$\begin{aligned} z^1_\xi &= \frac{V}{V^2 - a^2}\, z^1_\tau - \frac{1/\rho}{V^2 - a^2}\, z^2_\tau, \\ z^2_\xi &= -\frac{k}{V^2 - a^2}\, z^1_\tau + \frac{V}{V^2 - a^2}\, z^2_\tau, \end{aligned} \tag{6}$$

where $a = \sqrt{k/\rho}$ is the phase velocity of waves in the material.

Assuming that ρ, k are both ξ-dependent, we apply the Laplace transform in τ

$$\bar{z}(\xi, p) = \int_0^\infty e^{-p\tau} z(\xi, \tau) d\tau. \tag{7}$$

Eqs. (6) then obtain the form

$$\begin{aligned} \bar{z}^1_\xi &- \frac{p}{V^2 - a^2}\left(V\bar{z}^1 - (1/\rho)\bar{z}^2\right) = 0, \\ \bar{z}^2_\xi &+ \frac{p}{V^2 - a^2}\left(k\bar{z}^1 - V\bar{z}^2\right) = 0, \end{aligned} \tag{8}$$

with the coefficients periodic in ξ with period d. The Floquet analysis applied to this system reveals the following characterization of its solution.

Assume that $\xi \geq 0$, and material 1 occupies the intervals

$$(n - m_1)d \leq \xi \leq nd, \qquad n = 1, 2, \ldots, \tag{9}$$

whereas material 2 is concentrated within the supplementary intervals

$$nd \leq \xi \leq (n + m_2)d, \quad n = 0, 1, \ldots \tag{10}$$

Here m_1 and m_2 denote, respectively, the volume fractions of materials 1 and 2 in the lamination; clearly, $m_1 + m_2 = 1$.

A general solution to the system (8) is given by

$$\begin{aligned} \bar{z}^1 &= A_1 e^{\mu_1 \xi} P(\mu_1, \xi) + A_2 e^{\mu_2 \xi} P(\mu_2, \xi), \\ \bar{z}^2 &= A_1 e^{\mu_1 \xi} Q(\mu_1, \xi) + A_2 e^{\mu_2 \xi} Q(\mu_2, \xi), \end{aligned} \tag{11}$$

with $P_1, \ldots, Q_2$ being d-periodic functions.

Here, A_1 and A_2 denote the coefficients to be determined by the boundary conditions, and μ_1, μ_2 represent the Floquet characteristic exponents defined by the formula

$$\mu_{1,2}d = V(\theta_1/a_1 + \theta_2/a_2) \pm \chi(\theta_1, \theta_2) \tag{12}$$

with the upper (lower) sign related to $\mu_1(\mu_2)$, and

$$\begin{aligned} \theta_i &= pd\varphi_i, \ \varphi_i = m_i a_i/(V^2 - a_i^2), \ i = 1, 2, \\ & ch\chi(\theta_1, \theta_2) = ch\theta_1 ch\theta_2 + \sigma sh\theta_1 sh\theta_2, \\ \sigma &= (\gamma_1^2 + \gamma_2^2)/2\gamma_1\gamma_2, \ \gamma_i = k_i/a_i = \rho_i a_i = \sqrt{k_i \rho_i}, \ \ i = 1, 2. \end{aligned} \tag{13}$$

Clearly, $\sigma \geq 1$. When $q = pd/a_1 << 1$, Eq. (13) defines χ as

$$\chi = \sqrt{\theta_1^2 + \theta_2^2 + 2\sigma\theta_1\theta_2} = q\phi, \quad \phi = a_1\sqrt{\varphi_1^2 + \varphi_2^2 + 2\sigma\varphi_1\varphi_2}. \tag{14}$$

By (3), φ_1 and φ_2 are of the same sign; because $\sigma \geq 1$, the factor ϕ in (14) is real.

If $p = i\omega$ with ω real, then

$$ch\chi = \cos\omega d\varphi_1 \cos\omega d\varphi_2 - \sigma \sin\omega d\varphi_1 \sin\omega d\varphi_2.$$

If the absolute value of the right hand side of this equation exceeds 1, then the roots χ become real, and solution (11) contains exponentially increasing terms. This will happen if the value $\omega d/a_1$ falls into the relevant non-passing bands; the small values $\omega d/a_1 << 1$ do not belong to such bands, and the corresponding χ will be imaginary. The functions $P(\mu,\xi), Q(\mu,\xi)$ in (11) are given by the formulae

$$P(\mu,\xi) = \begin{cases} e^{-\left(\mu - \frac{p}{V-a_1}\right)(\xi - nd)} + Be^{-\left(\mu - \frac{p}{V+a_1}\right)(\xi - nd)}, & \xi \in \ (9), \\ \\ Ce^{-\left(\mu - \frac{p}{V-a_2}\right)(\xi - nd)} + De^{-\left(\mu - \frac{p}{V+a_2}\right)(\xi - nd)}, & \xi \in \ (10), \end{cases} \tag{15}$$

$$Q(\mu,\xi) = \begin{cases} \gamma_1 \left[-e^{-\left(\mu - \frac{p}{V-a_1}\right)(\xi - nd)} + Be^{-\left(\mu - \frac{p}{V+a_1}\right)(\xi - nd)} \right], & \xi \in \ (9), \\ \\ \gamma_2 \left[-Ce^{-\left(\mu - \frac{p}{V-a_1}\right)(\xi - nd)} + De^{-\left(\mu - \frac{p}{V+a_2}\right)(\xi - nd)} \right], & \xi \in \ (10). \end{cases} \tag{16}$$

Here, μ takes the values μ_1, μ_2, and $B = B(\mu)$, $C = C(\mu)$, and $D = D(\mu)$ are defined as solutions to the system

$$\begin{aligned} -B \ &+ \ C + D = 1 \\ B \ &+ \ (C - D)(\gamma_2/\gamma_1) = 1, \\ -Be^{\theta_1} \ &+ \ Ce^{\theta_2 \mp \chi} + De^{-\theta_2 \mp \chi} = e^{-\theta_1}, \end{aligned} \tag{17}$$

with upper (lower)sign related to $\mu = \mu_1$ and to $\mu = \mu_2$.

Both $P(\mu,\xi), Q(\mu,\xi)$ are d-periodic in ξ; these functions in fact depend on $\xi - nd$, this argument belonging to the range $[-m_1 d, 0]$ for (9), and to the range $[0, m_2 d]$ for (10).

$$-m_1 \leq \frac{\xi - nd}{d} \leq 0 \ \text{ for (9)}; \ \ 0 \leq \frac{\xi - nd}{d} \leq m_2 \text{ for (10)}.$$

In both cases, the difference $\xi - nd$ will be of order d. We may interpret (11) as modulated waves, with $e^{\mu\xi}$ being the long wave modulation factor, and $P(\mu,\xi), Q(\mu,\xi)$ representing the short wave carriers. The homogenization (averaging) procedure detects the low frequency envelopes $e^{\mu\xi}$ and eliminates the high frequency carriers P and Q.

3. THE LONG WAVE ASYMPTOTICS

In [1,2], there was obtained an asymptotic solution to this problem valid for an activated infinite elastic bar under the assumption $p = i\omega, \omega d/a_1 << 1$, i.e. for a low frequency dynamic disturbance. This solution was shown to satisfy a homogenized equation (1), i.e.

$$\left(z^1_x \frac{V^2\tilde{\rho}\left(\tilde{\frac{1}{k}}\right) - 1}{V^2\tilde{\rho} - \tilde{k}} \right)_x + V \left(z^1_x \frac{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right) - \left(\tilde{\frac{1}{a^2}}\right)}{V^2\tilde{\rho} - \tilde{k}} \right)_t$$

$$+V \left(z^1_t \frac{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right) - \left(\tilde{\frac{1}{a^2}}\right)}{V^2\tilde{\rho} - \tilde{k}} \right)_x - \left(z^1_t \frac{V^2 - \tilde{k}\left(\tilde{\frac{1}{\rho}}\right)}{V^2\tilde{\rho} - \tilde{k}} \right)_t \frac{1}{a_1^2 a_2^2} = 0. \quad (18)$$

Here, the symbol z^1 is preserved to denote the weak limit of the same quantity attained as $d \to 0$, i.e. the value of z^1 averaged over the period d of a laminate structure; with the aid of the volume fractions m_1 and m_2 of materials 1 and 2 in the layout we define the symbol $\tilde{\rho} = m_1\rho_2 + m_2\rho_1$, and so on. The symbols x, t in (18) are related to *slow* variables (versus fast variables $x/\epsilon, t/\epsilon$).

Eq. (18) governs the propagation of the envelopes of the modulated waves (11). It was obtained in [1] by a regular technique of homogenization, and in [2] with the aid of the Floquet Theory. The Floquet exponents were computed for Eqs. (8) in the low frequency limit; they are specified as (cf. (12))

$$\mu_{1,2} = \frac{p}{(V^2 - a_1^2)(V^2 - a_2^2)}$$

$$\left\{ V(V^2 - \tilde{a}^2) \pm a_1 a_2 \sqrt{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)\left(V^2 - \frac{\tilde{k}}{\tilde{\rho}}\right)\left(V^2 - \frac{\left(\tilde{\frac{1}{\rho}}\right)}{\left(\tilde{\frac{1}{k}}\right)}\right)} \right\}. \quad (19)$$

The second term in this formula corresponds to χ/d in Eq. (12); it is real once (3) holds.

A direct inspection shows that the ratios $p/\mu_i, i = 1, 2$, are of opposite signs if either (i) $V^2 < a_1^2 < a_2^2$, or (ii) $V^2 > a_2^2 > a_1^2$ and *simultaneously* $V^2 < \tilde{k}\left(\tilde{\frac{1}{\rho}}\right)$. If, however, $V^2 > a_2^2 > a_1^2$ but $V^2 > \tilde{k}\left(\tilde{\frac{1}{\rho}}\right)$, then $p/\mu_i,\ i = 1, 2$, appear to be of the same sign.

If $Re\ p = 0$, then the quantities $-p/\mu_i, i = 1, 2$, denote the phase velocities of the envelopes $e^{p\tau + \mu_i \xi}$ that emerge after we average the functions (11) over the period d of lamination. These velocities are measured in the coordinate frame $\xi = x - Vt,\ \tau = t$, moving at the speed V with respect to the laboratory frame (x,t). This motion occurs from left to right if $V>0$, and in the opposite direction otherwise. In a laboratory frame, however, the envelopes are specified as $e^{(p-\mu_i V)t+\mu_i x}$, with the phase velocities $-p/\mu_i+V$. In the low frequency limit, these velocities appear to be

$$-p/\mu_{1,2}+V=-\frac{V\left[\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)-\left(\tilde{\frac{1}{a^2}}\right)\right]\mp\frac{1}{a_1a_2}\sqrt{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)\left(V^2-\frac{\tilde{k}}{\tilde{\rho}}\right)\left(V^2-\frac{\left(\tilde{\frac{1}{\rho}}\right)}{\left(\tilde{\frac{1}{k}}\right)}\right)}}{\frac{1}{a_1^2a_2^2}\left[V^2-\tilde{k}\left(\tilde{\frac{1}{\rho}}\right)\right]},\tag{20}$$

with their product equal to

$$-a_1^2a_2^2\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)\frac{V^2-\frac{1}{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)}}{V^2-\tilde{k}\left(\tilde{\frac{1}{\rho}}\right)}.$$

The product is negative once

$$k_2>k_1,\ \rho_2<\rho_1 \text{ (regular mode)},\tag{21}$$

and it may be made positive by a suitable choice of V and m_1 if either

$$k_2>k_1,\rho_2>\rho_1,\ \text{or } k_2<k_1,\rho_2<\rho_1 \text{ (irregular mode)}.\tag{22}$$

We assume, however, that $a_2=\sqrt{k_2/\rho_2}>a_1=\sqrt{k_1/\rho_1}$ in all cases. We conclude that, for the irregular mode, a *coordinated* wave propagation may occur with respect to the (x,t)-frame, i.e. the envelopes may propagate in the same x direction. Since always

$$\tilde{k}\left(\tilde{\frac{1}{\rho}}\right)>\frac{1}{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)},$$

the coordinated wave propagation will take place if the velocity V may be so chosen that

$$\tilde{k}\left(\tilde{\frac{1}{\rho}}\right)>V^2>\frac{1}{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)},$$

and at the same time this velocity remains consistent with (3). This latter requirement can be satisifed only for the irregular mode (22). For example, if $k_2 = 10, \rho_2 = 9, k_1 = \rho_1 = 1$ and $1/72 < m_2 < 71/72$, then

$$\frac{1}{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)} < a_1^2 < a_2^2 < \tilde{k}\left(\tilde{\frac{1}{\rho}}\right), \tag{23}$$

and the coordinated waves become possible if either

$$\frac{1}{\tilde{\rho}\left(\tilde{\frac{1}{k}}\right)} < V^2 < a_1^2, \tag{24}$$

or

$$a_2^2 < V^2 < \tilde{k}\left(\tilde{\frac{1}{\rho}}\right). \tag{25}$$

We shall be particularly interested in the behaviour of solution in the irregular case (24) that holds when $k_2 > k_1, \rho_2 > \rho_1$ (see (22)). For this case, the phase velocities $-p/\mu_{1,2} + V$ (see (20)) are both positive if $V > 0$, and both negative if $V < 0$. In the latter case, the envelopes $e^{(p-\mu_i V)t+\mu_i x}$ both propagate towards the left end $x = 0$; on the other hand, at each instant t, there is an interval (9) or (10) adjacent to this end, and the original waves $e^{p\left(t+\frac{\xi}{V-a_1}\right)} = e^{p\frac{x-a_1 t}{V-a_1}}$ and $e^{p\left(t+\frac{\xi}{V-a_2}\right)} = e^{p\frac{x-a_2 t}{V-a_2}}$ propagate with the phase velocities a_1 or a_2 through these intervals from left to right, away from the point $x = 0$. These waves carry disturbances initiated by the boundary signal (4); these disturbances are partly reflected, partly transmitted at each encounter with the oncoming interfaces. Ultimately, the energy of these waves plus the energy pumped into (out of) the system by the external agent activating the material pattern, is transformed into the energy of low frequency (envelope) waves, and these waves leave the system through its left end $x = 0$. In the following section, we examine the asymptotics of solution valid in the vicinity of the point $x = 0$ as well as at the large distances from it. We shall see that the high frequency waves form up a wavefront propagating away from $x = 0$; the intensity of this wavefront decays exponentially with x.

4. THE WAVEFRONT: CASE $V < 0, V^2 < A_1^2 < A_2^2$

The solution $z^1(\xi, \tau)$ of (6) is taken to be

$$z^1(\xi, \tau) = \frac{1}{2\pi i}\int_{\mathcal{K}} \bar{z}^1 e^{p\tau}\, dp \tag{26}$$

where $\mathcal{K}$ is a path $Re\ p = \nu > 0$ to the right of all the singularities of the integrand (11) in the complex p-plane. The terms in (11) introduce the integrals

$$I_i = \frac{1}{2\pi i}\int_{\mathcal{K}} A_i(p)e^{p\tau+\mu_i\xi}P(\mu_i,\xi)dp, \quad i=1,2. \tag{27}$$

Suppose that $A_i(p)$ are regular for $Re\ p > \nu$, and take the large values of ν. Taking $V<0, V^2<a_1^2<a_2^2$, we obtain asymptotically (see (13))

$$e^{\chi} \sim \frac{1+\sigma}{2}e^{-(\theta_1+\theta_2)},$$

the exponentials $e^{\mu_i d}, i=1,2$ are calculated as

$$e^{\mu_1 d} \sim \frac{1+\sigma}{2}e^{pd\left(\frac{m_1}{V+a_1}+\frac{m_2}{V+a_2}\right)}, \tag{28}$$

$$e^{\mu_2 d} \sim \frac{2}{1+\sigma}e^{pd\left(\frac{m_1}{V-a_1}+\frac{m_2}{V-a_2}\right)}. \tag{29}$$

By (15), the product $e^{p\tau+\mu\xi}P(\mu,\xi) = e^{p\tau+\mu nd+\mu(\xi-nd)}P(\mu,\xi)$ appears to be

$$\begin{aligned} &\left[e^{\frac{p}{V-a_1}(\xi-nd)}+\bar{B}e^{\frac{p}{V+a_1}(\xi-nd)}\right]e^{p\tau+\mu nd}, \quad \xi\in(9),\\ &\left[\bar{C}e^{\frac{p}{V-a_2}(\xi-nd)}+\bar{D}e^{\frac{p}{V+a_2}(\xi-nd)}\right]e^{p\tau+\mu nd}, \quad \xi\in(10). \end{aligned} \tag{30}$$

For large $\nu>0$ and $\mu=\mu_2$, the system (17) specifies $B(\mu_2), C(\mu_2)$, and $D(\mu_2)$ as

$$B(\mu_2)\sim\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}, \quad C(\mu_2)\sim\frac{2\gamma_1}{\gamma_1+\gamma_2}, \quad D(\mu_2)\sim 0,$$

and the expressions (30) become

$$\begin{aligned} &\frac{2^n}{(1+\sigma)^n}\left[e^{\frac{p}{V-a_1}(\xi-nd)}+\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}e^{\frac{p}{V+a_1}(\xi-nd)}\right]\\ &\times\ e^{p\left[\tau+\xi\langle\frac{1}{V-a}\rangle\right]}e^{-p(\xi-nd)\langle\frac{1}{V-a}\rangle}, \quad \xi\in(9),\\ &\frac{2\gamma_1}{\gamma_1+\gamma_2}\,\frac{2^n}{(1+\sigma)^n}e^{\frac{p}{V-a_2}(\xi-nd)}e^{p\left[\tau+\xi\langle\frac{1}{V-a}\rangle\right]}\\ &\times\ e^{-p(\xi-nd)\langle\frac{1}{V-a}\rangle}, \quad \xi\in(10). \end{aligned} \tag{31}$$

If $p \to \infty, d \to 0$ but $pd/a_1 \to 0$, these formulas introduce a substantial factor $e^{p\left[\tau+\xi\langle \frac{1}{V-a}\rangle\right]}$; if $\tau + \xi\langle\frac{1}{V-a}\rangle < 0$, the contour $\mathcal{K}$ in (27) can be closed by a large semicircle to the right, and the integral (27) for $\mu = \mu_2$ will be zero:

$$I_2 = 0 \quad \text{if} \quad \tau + \xi\langle\frac{1}{V-a}\rangle < 0,$$

or, in the x, t variables,

$$I_2 = 0 \quad \text{if} \quad x - \left[V - \frac{1}{\langle\frac{1}{V-a}\rangle}\right] t > 0. \tag{32}$$

This integral defines the wave with the wavefront moving with velocity

$$w = V - \frac{1}{\langle\frac{1}{V-a}\rangle} = \frac{V\langle a\rangle - a_1 a_2}{V - \tilde{a}}. \tag{33}$$

Since $V < 0$, this velocity is positive, and the front propagates to the right. The difference $w - \frac{a_1 a_2}{\tilde{a}}$ is positive if $V < 0$; the motion of the pattern to the left accelerates the wavefront moving to the right. As to I_1, this integral should be neglected by taking $A_1(p) = 0$ since otherwise the factor $e^{p\tau+\mu_1\xi}$ becomes unbounded at large ν because $V < 0$ and $V^2 < a_1^2 < a_2^2$.

We conclude that the problem with the boundary condition

$$z^1(0, \tau) = f(\tau), \quad \tau > 0 \tag{34}$$

applied at $\xi = 0$, allows for the solution (27), with $i = 2$ and $A_2(p)$ defined as

$$A_2(p) = \frac{1}{1+B}\int_0^\infty f(\tau)e^{-p\tau}\,d\tau.$$

The behaviour of solution near the wavefront is determined by Eqs. (31). Due to (9), (10), the factors $\xi - nd$ may be replaced by zero in the first approximation, and we obtain near the wavefront

$$z^1(\xi, \tau) \sim \left(\frac{2}{1+\sigma}\right)^n f\left(\tau - \frac{\xi}{w - V}\right), \quad \tau > \frac{\xi}{w - V}. \tag{35}$$

Since $\sigma > 1$, the amplitude of the wave approaches zero as the number n of layers increases.

5. THE MAIN (STATIONARY) DISTURBANCE

In this section, we examine the asymptotic behaviour of the integral I_2 in the limit $\eta = \tau a_1/d \to \infty$, assuming that $\xi/\tau a_1 = const.$

As in a similar problem treated in [6], introduce nondimensional quantities

$$q = pd/a_1, \quad \kappa = \xi/\tau a_1,$$

and the time scale

$$\theta = 2\pi/\omega$$

related to the disturbance $f(t)$ (see (34)). The integral I_2 becomes

$$I_2 = \frac{1}{2\pi i}\frac{a_1}{d}\int_{\mathcal{K}} A_2\left(q\frac{a_1}{d}\right) e^{(q+\kappa\mu_2 d)\eta} P(\mu_2, \kappa d\eta) dq \tag{36}$$

The factor μ_2 depends on q, this dependence is given by (12) where we choose the lower sign ($\mu = \mu_2$)

$$\mu_2 d = Vqa_1(\varphi_1/a_1 + \varphi_2/a_2) - \chi(qa_1\varphi_1, qa_1\varphi_2). \tag{37}$$

When $q << 1$, the χ-term is defined by (14).

Given the structure (15) of $P(\mu, \xi)$, it is relevant to apply the method of steepest descent to calculate (36) for large η. Because the factors $\xi - nd$ in the exponents (15) are of order d, we shall treat these exponents as constants in the first approximation. We shall also assume that the function $B(\mu_2)$ is regular on the path $\mathcal{L}$ of the steepest descent.

The main part of I_2 comes from the neighborhood of the stationary point $q = q^*$ at which

$$\frac{d}{dq}(q + \kappa\mu_2 d) = 0. \tag{38}$$

We then obtain

$$\begin{aligned} I_2 \sim\ & exp[\eta(q^* + \kappa d\mu_2(q^*))]\frac{1}{2\pi i}\frac{a_1}{d}\int_{\mathcal{L}} A_2(q\frac{a_1}{d})P(\mu_2(q), \kappa d\eta) \\ &\cdot exp\left[\frac{1}{2}\eta\kappa d\mu_2''(q^*)(q-q^*)^2\right] dq, \end{aligned} \tag{39}$$

where we applied expansion of $q + \kappa\mu_2 d$ up to the guadratic term in $q - q^*$.

We now return to the variables p, ξ, τ:

$$\begin{aligned} I_2 \sim\ & exp[\tau p^* + \xi\mu_2(p^*)]\frac{1}{2\pi i}\int_{\mathcal{L}} A_2(p)P(\mu_2, \xi) \\ & exp\left[\frac{1}{2}\xi\mu_2''(p^*)(p-p^*)^2\right] dp, \end{aligned} \tag{40}$$

with p^* being the function of ξ, τ determined by

$$\tau + \xi \mu_2'(p^*) = 0. \tag{41}$$

Eq. (40) shows the asymptotic behaviour of I_2 at $\tau a/d \to \infty$ with $\kappa = \xi/\tau a_1$ fixed. The exponential factor $esp[\tau p^* + \xi \mu_2(p^*)]$ appears to be a predominant part of (40); it is stationary in ξ if

$$\tau \frac{\partial p^*}{\partial \xi} + \mu_2(p^*) + \xi \mu_2'(p^*) \frac{\partial p^*}{\partial \xi} = 0,$$

or, given (41), if

$$\mu_2(p^*) = 0.$$

It is seen from Eqs. (12), (13) that $p^* = 0$ is the root of $\mu_2(p)$; the derivative $\mu_2'(p*) = \mu_2'(0)$ is given by the factor of p at the rhs of (19) where we should take the lower sign. But for $V < 0, V^2 < a_1^2 < a_2^2$, the said factor is positive, and $\mu_2'(0)$ is also positive. This means that (41) may hold only for $\xi < 0$, i.e. outside the admissible domain $\xi \leq 0$. We conclude that the envelope waves with phase velocities (20) cannot propagate in this case. The solution is reduced to the expression (35); the disturbance damps out to zero behind the wavefront as it propagates away from $\xi = 0$.

6. THE ASYMPTOTICS IN (X, T)-VARIABLES

Eq. (35) represents the asymptotic solution expressed through the variables (ξ, τ). Returning to the original variables x, t (see (5)), we find the value $g(t)$ of z^1 at $x = 0$ to be equal to $z^1(-Vt, t)$ calculated from (35):

$$g(t) = z^1(-Vt, t) \sim \left(\frac{2}{1+\sigma}\right)^n f\left(\frac{w}{w - Vt}\, t\right). \tag{42}$$

If $x = 0$ then $\xi = -Vt$; for ξ belonging to the nth interval (9) or (10), we may apply the approximation $\xi \sim nd$, and, consequently,

$$n \sim -\frac{Vt}{d}.$$

Eq. (42) now takes the form

$$g(t) \sim \left(\frac{2}{1+\sigma}\right)^{-\frac{Vt}{d}} f\left(\frac{w}{w - V}\, t\right).$$

The function $f(\theta)$ now becomes

$$f(\theta) = \left(\frac{2}{1+\sigma}\right)^{\frac{V}{d}\ \frac{w-V}{w}\ \theta} g\left(\frac{w-V}{w}\ \theta\right).$$

Applying this towards (35) an defining n as $n \sim \xi/d$, we arrive at the asymptotic expression for $z^1(x,t)$:

$$z^1(x,t) \sim \left(\frac{2}{1+\sigma}\right)^{\frac{w-V}{dw}\ x} g\left(t - \frac{x}{w}\right).$$

By (33), both w and $w-V$ are positive if $V<0$; we observe that accomodation to the boundary condition $z^1(0,t) = g$ occurs through the boundary layer of thickness $wd/[(w-V)\ln(1+\sigma)/2]$.

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