Discussion of "An Analytic Expansion Method for Forward-Backward Stochastic Differential Equations"

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Probability based methods.

1. J. San Martin J. Ma, P. Protter and S. Torres. Numerical methods for Backward Stochastic Differential Equations. Annals of Applied Probability, 12(1):302316, 2002.
2. E. Gobet, J.P Lemor, and X. Warin. A regression based Monte Carlo method to solve Backward Stochastic Differential Equations. Annals of Applied Probability, 15(3):21722002, 2005.
3. B. Bouchard and N. Touzi. Discrete time approximation and Monte Carlo simulation for Backward Stochastic Differential Equations. Stochastic processes and their applications, 111:175206, 2004

PDE based method

The paper gives a method for approximating solutions to linear parabolic PDEs applied to BSDEs via the Feynman-Kac formula.

FBSDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d$$
$$dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = \psi(X_T) \in \mathbb{R}.$$

Feynman-Kac formula

Let u be the solution to the quasi-linear PDE:

$$\partial_t u(t,x)+\mathcal{A}u(t,x)+f(t,x,u(t,x),\nabla u(t,x)\cdot\sigma(t,x))=0, \quad u(T,\cdot)=\psi(\cdot), \tag{2}$$

where $\mathcal{A}$ is the generator of the X process. Then

$$Y_t = u(t,X_t) \qquad \text{and} \qquad Z_t = \nabla_x u(t,X_t)\cdot\sigma(t,X_t).$$

Picard Iteration

$$Y_t^{(0)} := E_t\psi(X_T) + E_t\int_t^T f(s, X_s, 0, 0)ds, \qquad Z_t^{(0)} := \mathcal{D}_t Y_t^{(0)},$$

and for $k \geq 1$,

$$dY_s^{(k)} = -f(s, X_s, Y_s^{(k-1}, Z_s^{(k-1)})ds + Z_s^{(k)}dW_s.$$

Then $(Y^{(k)}, Z^{(k)}) \to (Y, Z)$ in $L^2(dP \times dt)$.

Linear Parabolic PDE

$Y^{(k)} = u^{(k)}(t, X_t)$ and $Z^{(k)} = \nabla u^{(k)}(t, X_t) \cdot \sigma(t, X_t)$ where $u^{(k)}$ is the solution to the linear parabolic PDE

$$(\partial_t + \mathcal{A})u^{(k)} + f(t, x, u^{(k-1)}, \nabla u^{(k-1)} \cdot \sigma) = 0, \qquad u^{(k)}(T, \cdot) = \psi(\cdot). \quad (3)$$

Paper outlines an approximation scheme for $u^{(k)}$ (solution to linear parabolic PDE) based on the method by *Lorig, M., Pagliarani, S., and Pascucci, A. (2015a). "Analytical expansions for parabolic equations". SIAM Journal on Applied Mathematics, 75:468491*

Approximation scheme

Fix a point $\bar{x} \in \mathbb{R}^d$ and use a Taylor series expansion of coefficients of $\mathcal{A}$ about the point $\bar{x}$. So

$$\mathcal{A} = \mathcal{A}_0^{\bar{x}} + \mathcal{A}_1^{\bar{x}} + \mathcal{A}_2^{\bar{x}} + \dots.$$

$\mathcal{A}_0^{\bar{x}}$ is an elliptic operator with constant coefficients, $\mathcal{A}_i^{\bar{x}}$, $i \geq 1$ are differential operators with polynomial coefficients.

Assume $u = \sum_{i=0}^{\infty} u_i^{\bar{x}}$, then u is a solution to (3) if

$$(\partial_t + \mathcal{A}_0)u_0^{\bar{x}} + f = 0, \quad u_0^{\bar{x}}(T, \cdot) = \psi(\cdot) \tag{4a}$$

$$(\partial_t + \mathcal{A}_0)u_l^{\bar{x}} + \sum_{i=1}^{l} \mathcal{A}_i^{\bar{x}} u_{l-i}^{\bar{x}} = 0, \quad u_l^{\bar{x}}(T, \cdot) = 0. \tag{4b}$$

The $u_l^{\bar{x}}$ can be solved recursively by above equations.
Observation: Calculations are possible as the Green's function for $\partial_t + \mathcal{A}_0^{\bar{x}}$ is the Gaussian kernel!

Solving for u_l:

$$u_0^{\bar{x}}(t,x) = \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;T,y)\psi(y)dy + \int_t^T \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;t_1,y)f(t_1,y)dydt_1$$

$$u_l^{\bar{x}}(t,x) = \int_t^T \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;t_1,y) \left(\sum_{i=1}^{l} \mathcal{A}_i^{\bar{x}} u_{l-i}^{\bar{x}}(t_1,y) \right) dydt_1.$$

For explicit formulas, approximate source terms and terminal condition by polynomials (Taylor expansion about $\bar{x}$).

Error bounds - Gaussian moments

If approximate ψ by its Taylor expansion $T_m^{\bar{x}}\psi$ of order m, then

$$|\psi(x) - T_m^{\bar{x}}\psi(x)| \leq C|x - \bar{x}|^{m+1},$$

and so the error due to this approximation is of order

$$\int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t, x; T, y)|y - \bar{x}|^{m+1} dy|_{\bar{x}=x} \leq c(T - t)^{\frac{m+1}{2}}.$$

Error is small if $(T - t)$ is small.

Discretize time

Let $\{t_i\}_{i=1}^{n}$ be a partition of $[0, T]$ such that $t_{i+1} - t_i = T/n$. (Let's just look at the first order approximation u_0). Then

$$u_{0,m,n}^{\bar{x}}(t,x) = \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;T,y) T_m^{\bar{x}} \psi(y) dy + \int_t^T \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;t_1,y) T_m^{\bar{x}} f(t_1,y) dy dt_1 \qquad \text{for } t \in [t_{n-1}, T).$$

For $t \in [t_{n-2}, t_{n-1})$, solve the PDE with the terminal condition $u^{tc}(t_{n-1}, x) = u_{0,m,n}^{\bar{x}}(t_{n-1}, x)|_{\bar{x}=x}$:

$$u_{0,m,n}^{\bar{x}}(t,x) = \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;t_{n-1},y) T_m^{\bar{x}} u^{tc}(t_{n-1},y) dy + \int_t^T \int_{\mathbb{R}^d} \Gamma_0^{\bar{x}}(t,x;t_1,y) T_m^{\bar{x}} f(t_1,y) dy dt_1 \qquad \text{for } t \in [t_{n-2}, t_{n-1}).$$

Thus errors will now be of order $\left(\frac{T}{n}\right)^{power}$.

Important Features

- Several layers of approximation (Picard iteration, Taylor expansion of coefficients of $\mathcal{A}$, Taylor expansion of terminal/source term, Time discretization) makes the computation of error bounds complicated. But implementing the method is "simple" due to the recursive nature of the computations.
- Error bounds given
- Method depends on the existence of the Feynman-Kac formula
- For better approximations need the coefficients to be sufficiently regular

Questions

- How does this compare to other approximation methods for parabolic PDEs? For example:
  Approximate solutions to second order parabolic equations. I: Analytic estimates, Constantinescu, Radu and Costanzino, Nick and Mazzucato, Anna L. and Nistor, Victor, Journal of Mathematical Physics, 51, 103502 (2010).
- Can this method be extended to BSDEs with jumps? Can this method be applied to partial integro-differential equations?
- In the Numerical example (pricing a European put option), the terminal condition is bounded and continuous but not differentiable. A smooth approximation of the payoff function (terminal condition) was used.
  Can this method be extended to cases where the driver and terminal condition are not differentiable functions?
- Error bound:

$$||Y_{\cdot} - Y_{\cdot}^{(k,l,m,n)}||^2_{L^2(dP\times dt)} \leq K\frac{(2\delta)^k}{1-2\delta} + C\left(\frac{T-t}{n}\right)^{l+2} + ...,$$

where $\delta \in (0, \frac{1}{2})$.