# An Analytical Expansion Method for FBSDEs
# 1st Eastern Conference on Mathematical Finance

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# Introduction

1. This paper discusses one numerical scheme to solve general uncoupled FBSDEs.
2. An FBSDE can be written as

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \qquad X_0 = x_0, \tag{1}$$

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \qquad Y_T = \psi(X_T). \tag{2}$$

3. Pair $(Y, Z)$ is called the solution to the FBSDE system (1)-(2).

# The Numerical Scheme of DLRSZ2015: Picard Iteration

1. First, set $(Y^{(0)}, Z^{(0)}) := (0, 0)$.
2. Second, for general $k \geq 1$, we define recursively $(Y^{(k)}, Z^{(k)})$ as the solution to

$$\mathrm{d}Y_t^{(k)} = -f(t, X_t, Y_t^{(k-1)}, Z_t^{(k-1)})\mathrm{d}t + Z_t^{(k)}\mathrm{d}W_t, \quad (3)$$

$$Y_T^{(k)} = \psi(X_T), \quad (4)$$

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad (5)$$

$$X_t = x_0. \quad (6)$$

3. Under some technical conditions, we have

$$\|Y^{(k)} - Y\|_\eta^2 + \|Z^{(k)} - Z\|_\eta^2 \leq K\epsilon^k, \quad (7)$$

$$\|\xi\|_\eta^2 = \mathbb{E}\left[\int_0^T \exp(\eta t)|\xi_t|^2 \mathrm{d}t\right]. \quad (8)$$

# The Numerical Scheme of DLRSZ2015: From Linear FBSDE to Linear Parabolic PDE

The linear FBSDE at $k$-th iteration is related to a linear parabolic PDE

$$
\begin{aligned}
0 &= \partial_t u^{(k)} + \mu(t,x)\partial_x u^{(k)} + \frac{1}{2}\mathbf{tr}[\sigma(t,x)\sigma(t,x)^\intercal \partial_x^2 u^{(k)}] && (9)\\
&\quad + f(t,x,u^{(k-1)}(t,x), \partial_x u^{(k-1)}(t,x)\sigma(t,x)), && (10)\\
\psi(x) &= u^{(k)}(T,x). && (11)
\end{aligned}
$$

We will drop the super-index and write

$$
\begin{aligned}
0 &= \partial_t u + \mu(t,x)\partial_x u + \frac{1}{2}\mathbf{tr}[\sigma(t,x)\sigma(t,x)^\intercal \partial_x^2 u] && (12)\\
&\quad + f(t,x), && (13)\\
\psi(x) &= u(T,x). && (14)
\end{aligned}
$$

# The Numerical Scheme of DLRSZ2015: Solving Linear Parabolic PDEs

1. If we can provide a numerical scheme to the linear parabolic PDE obtained at every iteration, then we can insert the solution into the Picard iteration process and obtain the approximate solution to the FBSDE system.
2. We follow and extend Lorig et al (2014) paper on parabolic PDE expansion to compute the approximate solution to the linear parabolic PDE.

# The Numerical Scheme of DLRSZ2015: Lorig et al 2014's Arguments

1. Write the infinitesimal generator of the Markov process $X$

$$\mathcal{A} = \sum_{i=1}^{d} \mu_i(t,x)\partial_{x_i} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma\sigma^{\mathsf{T}})_{i,j}(t,x)\partial_{x_i}\partial_{x_j}, \tag{15}$$

$$\tag{16}$$

2. Write

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), \quad |\alpha| = \sum_{i=1}^{d} \alpha_i, \quad \partial_x^\alpha = \prod_{i=1}^{d} \partial_{x_i}^{\alpha_i}, \tag{17}$$

$$x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}, \qquad \alpha! = \prod_{i=1}^{d} \alpha_i!. \tag{18}$$

# The Numerical Scheme of DLRSZ2015: Lorig et al 2014's Arguments

1. Write

$$\mathcal{A} = \sum_{i=0}^{\infty} \mathcal{A}_i^{\bar{x}}, \tag{19}$$

$$\mathcal{A}_i^{\bar{x}} = \sum_{1 \le |\alpha| \le 2} a_{\alpha,i}^{\bar{x}}(t,x)\partial_x^{\alpha}, \tag{20}$$

$$a_{\alpha,i}^{\bar{x}}(t,x) = \sum_{|\beta|=i} \frac{1}{\beta!}\partial_x^{\beta} a_{\alpha}(t,\bar{x})(x-\bar{x})^{\beta}. \tag{21}$$

2. Write

$$u = \sum_{l=0}^{\infty} u_l^{\bar{x}}. \tag{22}$$

# The Numerical Scheme of DLRSZ2015: Lorig et al 2014's Arguments

1. The function $u_l^{\bar{x}}$ for $l = 0, 1, 2, \cdots$ solves

$$\left.\begin{aligned} \left(\partial_t + \mathcal{A}_0^{\bar{x}}\right) u_0^{\bar{x}} + f = 0, \quad u_0(T, \cdot) = \psi, \\ \left(\partial_t + \mathcal{A}_0^{\bar{x}}\right) u_l^{\bar{x}} + \sum_{i=1}^{l} \mathcal{A}_i^{\bar{x}} u_{l-i}^{\bar{x}} = 0, \quad u_l^{\bar{x}}(T, \cdot) = 0, \quad l \geq 1. \end{aligned}\right\} \tag{23}$$

2. The solution of $u_l^{\bar{x}}$ depends on $u_j^{\bar{x}}$ for $j = 0, 1, 2, \cdots, l-1$.

# The Numerical Scheme of DLRSZ2015: Lorig et al 2014's Arguments

1. However, unless the coefficients of the FBSDE satisfy some technical conditions on the explosion speed of the sup-norm of their higher order derivatives, it is not guaranteed that

$$u = \sum_{l=0}^{\infty} u_l^{\bar{x}}, \tag{24}$$

   is finite or even exists. So, the algorithm is not guaranteed to converge in $l$-direction.

2. Also, the PDE appearing above is hard to compute.
3. In order to obtain the convergence and applicability, we have to introduce other structures: time discretization and Taylor expansion.

# The Numerical Scheme of DLRSZ2015: DLRSZ2015 Arguments

1. Divide the interval $[t, T]$ into $n$ equally spaced intervals $[t_{i-1}, t_i]$ with $i = 1, 2, \ldots, n$, where

$$t_i = t + i\delta_t, \quad \delta_t = (T - t)/n, \quad i = 0, 1, 2, \ldots, n. \tag{25}$$

2. We define $u^{\bar{x}}_{l,m,n}$ as the solution of the following sequence of PDEs for $t \in [t_{n-1}, T)$

$$\left.\begin{aligned} \left(\partial_t + \mathcal{A}^{\bar{x}}_0\right) u^{\bar{x}}_{0,m,n} + \mathbf{T}^{\bar{x}}_m f = 0, \quad u^{\bar{x}}_{0,m,n}(T, \cdot) = \mathbf{T}^{\bar{x}}_m \psi, \\ \left(\partial_t + \mathcal{A}^{\bar{x}}_0\right) u^{\bar{x}}_{l,m,n} + \sum_{i=1}^{l} \mathcal{A}^{\bar{x}}_i u^{\bar{x}}_{l-i,m,n} = 0, \quad u^{\bar{x}}_{l,m,n}(T, \cdot) = 0, \end{aligned}\right\} \tag{26}$$

# The Numerical Scheme of DLRSZ2015: DLRSZ2015 Arguments

1. For $t \in [t_{n-j-1}, t_{n-j})$, we define

$$\left.\begin{aligned} \left(\partial_t + \mathcal{A}_0^{\bar{x}}\right) u_{0,m,n}^{\bar{x}} + \mathbf{T}_m^{\bar{x}} f &= 0, \\ u_{0,m,n}^{\bar{x}}(t_{n-j}, \cdot) &= \mathbf{T}_m^{\bar{x}} u_{0,m,n}^{x}(t_{n-j}, \cdot), \\ \left(\partial_t + \mathcal{A}_0^{\bar{x}}\right) u_{l,m,n}^{\bar{x}} + \sum_{i=1}^{l} \mathcal{A}_i^{\bar{x}} u_{l-i,m,n}^{\bar{x}} &= 0, \\ u_{l,m,n}^{\bar{x}}(t_{n-j}, \cdot) &= \mathbf{T}_{m-2l}^{\bar{x}} u_{l,m,n}^{x}(t_{n-j}, \cdot), \quad \geq 1, \end{aligned}\right\} \tag{27}$$

# The Numerical Scheme of DLRSZ2015: Some Remarks

1. The complexity of the structure of the expansions solution
2. Strong requirements on the coefficients of the FBSDEs
3. Long maturity problems
4. Potentially low convergence speed for *fast varying functions*
5. Presently unable to handle FBSDEs with Lévy jumps or Coupled FBSDEs
6. Many more types of SDEs, FBSDEs and PIDEs
    1. Mckean-SDEs
    2. Reflected doubly FBSDEs, constrained FBSDEs
    3. Cauchy-Dirichlet, Cauchy-Neumann, Cauchy-Robin problems
7. *Curse of dimensionality*