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Market Stability and Indifference Prices

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Based on
"Stability of Utility Maximization in Nonequivalent Markets,"
Finance & Stochastics (2016)

Basic Problem

Consider a financial market consisting of

- Bank account with $r = 0$
- Traded risky asset S
- Derivative security $\phi(S')$ with non-traded underlying S' with

$$\mathrm{corr}(S', S) \approx 1$$

Question:

$$\mathrm{corr}(S', S) \longrightarrow 1 \overset{?}{\Longrightarrow} \mathrm{price}(\phi(S')) \longrightarrow \mathrm{price}(\phi(S))?$$

Examples:

- Derivatives on oil
- Hedging with an index

Related Work

- Exponential investor where S, S' are correlated geometric Brownian motions:
  - Davis (working paper 2000) - value function
  - Monoyois (QF 2004) - indifference prices
- Utility Maximization Stability Literature:
  - Larsen-Žitković (SPA 2007)
  - Bayraktar-Kravitz (SPA 2013)
  - Kardaras-Žitković (MF 2011)
- Main hurdle: All previous stability work crucially relies on a fixed volatility structure.

Counterexample (I) – Set-up

Let B and W be two independent Brownian motions.

Financial Market:

- Bank account with $r = 0$
- Stock (indexed by $\rho \in (-1,1)$)

$$dS_t^\rho = S_t^\rho \left(dt + \sqrt{1-\rho^2} dB_t + \rho dW_t \right), \quad S_0^\rho := 1$$

- Financial derivative with payoff $\phi(B_T)$:

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is bounded, continuous, not constant}$$

  Let $\phi_{\min} := \inf_{x \in \mathbb{R}} \phi(x)$
- Unless $\rho = 0$, the payoff $\phi(B_T)$ cannot be replicated.

Counterexample (II) – Utility Maximization Problem

Consider an investor with

- $x_0 > -\phi_{\min}$: Initial wealth
- $U(x) = \frac{x^p}{p}$: Utility function ($p < 1$, $p = 0$ corresponds to $\log$)

Objective:

$$u(x_0,\rho) := \sup_{H \in \mathcal{A}(\rho)} \mathbb{E}\left[U\left(x_0 + \int_0^T H dS^\rho + \phi(B_T) \right) \right],$$

where

$$\mathcal{A}(\rho) := \{H : \exists K = K(H),\ \int_0^t H dS^\rho \geq -K \ \forall t\}.$$

Question: Does $u(x_0,\rho)$ converge to $u(x_0,0)$ as $\rho \to 0$?

Counterexample (III) – Admissibility Constraints

Let $H \in \mathcal{A}(\rho)$ such that

$$x_0 + \int_0^T H dS^\rho + \phi(B_T) \geq 0.$$

Two Cases:

- $\rho = 0$: We can replicate $\phi(B_T) = p + \int_0^T \Delta dS^0$

$$x_0 + \int_0^T H dS^0 + \phi(B_T) = x_0 + p + \int_0^T \underbrace{(H + \Delta)}_{\widetilde{H}} dS^0 \geq 0$$

- $\rho \neq 0$: Cannot replicate $\phi(B_T)$ by trading in S^ρ. It can be shown (see El Karoui & Quenez, 1995) that

$$x_0 + \int_0^T H dS^\rho + \phi_{\min} \geq 0$$

Yet $\phi_{\min} = \inf_x \phi(x) < p$. Therefore, the $\rho \neq 0$ markets are *strictly more restrictive* than the $\rho = 0$ market.

Counterexample (IV) – Results

Theorem (W. 2016) $u(x_0,\rho)$ does not converge to $u(x_0,0)$ as $\rho \to 0$.

For $x_0 > 0$, consider the value function without random endowment

$$w(x_0,\rho) := \sup_{H \in \mathcal{A}(\rho)} \mathbb{E}\left[U\left(x_0 + \int_0^T H dS^\rho\right)\right].$$

We define the *utility indifference price* of $\phi(B_T)$ by $p = p(x_0,\rho)$ such that

$$u(x_0,\rho) = w(x_0 + p,\rho).$$

For $\rho = 0$, $p(x_0,0)$ is the unique arbitrage-free price.

Corollary (W. 2016) Indifference prices do not converge:

$$\limsup_{\rho \to 0} p(x_0,\rho) < p(x_0,0).$$

Positive Result (I)

Problems arise from the property that $U(x) = -\infty$ for $x < 0$.

- Real-line utility function: $U : \mathbb{R} \to \mathbb{R}$ (main example is $U(x) = -e^{-ax}$)
- In a Brownian framework, for $1 \leq n \leq \infty$, we consider

$$dS^n = dM^n + \lambda^n d\langle M^n \rangle, \ S_0^n \in \mathbb{R}.$$

  Suppose $S^n \longrightarrow S^\infty$ and $\int_0^\cdot \lambda^n dM^n \longrightarrow \int_0^\cdot \lambda^\infty dM^\infty$ in the semimartingale topology.

- Financial derivative payoff: $V_T \in L^\infty$.

Theorem (W. 2016) Under appropriate integrability and nondegeneracy conditions, the value functions and indifference prices converge as $n \to \infty$.

Positive Result (II)

Main Difficulty: Because of the changing volatility structure, small perturbations in the limiting market's investment strategies are not consistent with strategies in the pre-limiting markets.

- Primal Problem: Suppose $(H \cdot S^\infty)$ is S^∞-admissible.

  Is $(H \cdot S^n)$ S^n-admissible?
  Is H even S^n-integrable?

- Dual Problem: Consider a dual element $\mathcal{E}(-\lambda^\infty \cdot M^\infty)_T \mathcal{E}(L)_T$ where $\langle L, (\lambda^\infty \cdot M^\infty)\rangle \equiv 0$.

  But $\langle L, (\lambda^n \cdot M^n)\rangle = ?$

THANK YOU!