Sensitivity Analysis on Long-term Cash flows

Hyungbin Park
Worcester Polytechnic Institute

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Introduction

Related articles

- Borovicka, J., Hansen, L.P., Hendricks, M., Scheinkman, J.A.: Risk price dynamics. Journal of Financial Econometrics **9**(1), 3-65 (2011)
- Fournie, E., Lasry J., Lebuchoux, J., Lions P., Touzi, N.: Applications of Mallivin calculus to Monte Carlo methods in finance. Finance Stoch. **3,** 391-412 (1999)
- Hansen, L.P., Scheinkman, J.A.: Long-term risk: An operator approach. Econometrica **77,** 177-234 (2009)
- Hansen, L.P., Scheinkman, J.A.: Pricing growth-rate risk. Finance Stoch. **16**(1), 1-15 (2012)

Introduction

Let $W_t = (W_1(t), W_2(t), \cdots, W_d(t))^\top$ be a standard d-dimensional Brownian motion.

Assumption

An underlying process X_t is a conservative d-dimensional time-homogeneous Markov diffusion process with the following form:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t\ , \quad X_0 = \xi\ .$$

Here, b is a d-dimensional column vector and σ is a $d \times d$ matrix with some technical conditions.

Introduction

In finance, we often encounter the quantity of the form:

$$p_T := \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_t)\,dt} f(X_T)]\ .$$

Purpose: to study a **sensitivity analysis** for the quantity p_T with respect to the perturbation of X_t for large T.

This sensitivity is useful for long-term static investors and for long-dated option prices.

Introduction

Let X_t^ϵ be a perturbed process of X_t (with the same initial value $\xi = X_0 = X_0^\epsilon$) of the form:

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\,dt + \sigma_\epsilon(X_t^\epsilon)\,dW_t\ . \tag{1.1}$$

The perturbed quantity is given by

$$p_T^\epsilon := \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s^\epsilon)\,ds} f(X_T^\epsilon)]\ .$$

For the sensitivity analysis, we compute

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} p_T^\epsilon$$

and investigate the behavior of this quantity for large T.

Introduction

For the sensitivity w.r.t. the initial value $X_0 = \xi$, we compute

$$\frac{\partial p_T}{\partial \xi}$$

and investigate the behavior of this quantity for large T.

Martingale extraction

Martingale extraction

We denote the infinitesimal generator corresponding to the operator

$$f \mapsto p_T = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_t)\,dt} f(X_T)]$$

by $\mathcal{L}$:

$$\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} - r(x)$$

where $a = \sigma\sigma^\top$.

Martingale extraction

Let (λ, ϕ) be an eigenpair of $\mathcal{L}\phi = -\lambda\phi$ with positive function ϕ. It is easily checked that

$$M_t := e^{\lambda t - \int_0^t r(X_s)ds}\, \phi(X_t)\, \phi^{-1}(\xi)$$

is a local martingale.

Definition

When the local martingale M_t is a martingale, we say that (X_t, r) admits the *martingale extraction* with respect to (λ, ϕ).

$e^{-\int_0^t r(X_s)ds} = M_t\, e^{-\lambda t}\, \phi^{-1}(X_t)\, \phi(\xi)$
: martingale M_t is extracted from the discount factor

Martingale extraction

When M_t is a martingale, we can define a new measure $\mathbb{P}$ by $\mathbb{P}(A) := \int_A M_T \, d\mathbb{Q} = \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_A M_T]$ for $A \in \mathcal{F}_T$, that is,

$$M_T = \left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T}$$

The measure $\mathbb{P}$ is called *the transformed measure* from $\mathbb{Q}$ with respect to (λ, ϕ).

p_T can be expressed by

$$\begin{aligned} p_T &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s)ds} f(X_T)] \\ &= \phi(\xi)\, e^{-\lambda t} \cdot \mathbb{E}^{\mathbb{Q}}[M_t\, (\phi^{-1} f)(X_t)] \\ &= \phi(\xi)\, e^{-\lambda T} \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]\,. \end{aligned}$$

Martingale extraction

We have

$$p_T = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s)ds} f(X_T)] = \phi(\xi)\, e^{-\lambda T} \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]\,.$$

This relationship implies that the quantity p_T can be expressed in a relatively more manageable manner. The term $\mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]$ depends on the final value of X_T, whereas $\mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s)ds} f(X_T)]$ depends on the whole path of X_t at $0 \le t \le T$.

This advantage makes it easier to analyze the sensitivity of long-term cash flows.

Martingale extraction

In general, there are infinitely many ways to extract the martingale. We choose a special one.

Definition

Consider a martingale extraction such that $\mathbb{E}^{\mathbb{P}}[(\phi^{-1}f)(X_T)]$ converges to a nonzero constant as $T \to \infty$. We say this is a martingale extraction *stabilizing* f. In this case,

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\lambda \, .$$

For example, if X_t has an invariant distribution μ under $\mathbb{P}$, then

$$\mathbb{E}^{\mathbb{P}}[(\phi^{-1}f)(X_T)] \to \int (\phi^{-1}f)(z)\, d\mu(z)$$

for suitably nice f.

Sensitivity Analysis

Sensitivity Analysis

The rho and the vaga

Let X_t^ϵ be a perturbed process of X_t (with the same initial value $\xi = X_0 = X_0^\epsilon$) of the form:

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\,dt + \sigma_\epsilon(X_t^\epsilon)\,dW_t\;. \tag{3.1}$$

The perturbed quantity is given by

$$p_T^\epsilon := \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s^\epsilon)\,ds} f(X_T^\epsilon)]\;.$$

For the sensitivity analysis, we compute

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \ln p_T^\epsilon$$

and investigate the behavior of this quantity for large T.

Sensitivity Analysis

Assume that (X_t^ϵ, r) also admits the martingale extraction that stabilizes f, then

$$p_T^\epsilon = \phi_\epsilon(\xi)\, e^{-\lambda(\epsilon)T} \cdot \mathbb{E}^{\mathbb{P}_\epsilon}[(\phi_\epsilon^{-1} f)(X_T^\epsilon)]\,.$$

Differentiate with respect to ϵ and evaluate at $\epsilon = 0$, then

$$\begin{aligned}\frac{\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} p_T^\epsilon}{T \cdot p_T} &= \frac{\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \phi_\epsilon(\xi)}{T \cdot \phi(\xi)} - \lambda'(0) + \frac{\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \mathbb{E}^{\mathbb{P}}[(\phi_\epsilon^{-1} f)(X_T)]}{T \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]} \\ &\quad + \frac{\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \mathbb{E}^{\mathbb{P}_\epsilon}[(\phi^{-1} f)(X_T^\epsilon)]}{T \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)]}\,.\end{aligned}$$

Here, $\mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)] \to$ (a nonzero constant) as $T \to \infty$.

Sensitivity Analysis

The long-term behavior of the rho and the vaga

Under some conditions, the first, third and the last terms go to zero as $T \to \infty$, thus we have

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \ln p_T^{\epsilon} = \lim_{T\to\infty} \frac{\frac{\partial}{\partial \epsilon}\big|_{\epsilon=0} p_T^{\epsilon}}{T \cdot p_T} = -\lambda'(0)$$

Sensitivity Analysis

The dynamics under the transformed measure

Let

$$\varphi := \sigma^\top \cdot \frac{\nabla \phi}{\phi} \quad \text{(Girsanov kernel)}$$

then

$$B_t := W_t - \int_0^t \varphi(X_s)\, ds$$

is a Brownian motion under $\mathbb{P}$. The $\mathbb{P}$-dynamics of X_t are

$$\begin{aligned} dX_t &= b(X_t)\, dt + \sigma(X_t)\, dW_t \\ &= (b(X_t) + \sigma(X_t)\varphi(X_t))\, dt + \sigma(X_t)\, dB_t \ . \end{aligned}$$

Sensitivity Analysis

The rho

Want to control: $\frac{1}{T} \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \mathbb{E}^{\mathbb{P}_\epsilon}[(\phi^{-1} f)(X_T^\epsilon)] \to 0$ as $\to \infty$

The perturbed process X_t^ϵ expressed by

$$\begin{aligned} dX_t^\epsilon &= b_\epsilon(X_t^\epsilon)\, dt + \sigma(X_t^\epsilon)\, dW_t \\ &= (\sigma^{-1} b_\epsilon + \varphi_\epsilon)(X_t)\, dt + \sigma(X_t^\epsilon)\, dB_t^\epsilon \\ &= k_\epsilon(X_t^\epsilon)\, dt + \sigma(X_t^\epsilon)\, dB_t^\epsilon \ . \end{aligned}$$

Sensitivity Analysis

The rho

Assume that there exists a function $g : \mathbb{R}^d \to \mathbb{R}$ with

$$\left| \frac{\partial k_\epsilon(x)}{\partial \epsilon} \right| \leq g(x)$$

on $(\epsilon, x) \in I \times \mathbb{R}^d$ for an open interval I containing 0 such that

(i) there exists a positive number ϵ_0 such that

$$\mathbb{E}^{\mathbb{P}} \left[\exp\left(\epsilon_0 \int_0^T g^2(X_t)\, dt \right) \right] \leq c\, e^{aT}$$

for some constants a and c on $0 < T < \infty$.

(ii) for each $T > 0$, there is a positive number ϵ_1 such that $\mathbb{E}^{\mathbb{P}} \int_0^T g^{2+\epsilon_1}(X_t)\, dt$ is finite.

(iii) $\frac{1}{T} \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)^2(X_T)] \to 0$ as $T \to \infty$.

Sensitivity Analysis

Then, $\mathbb{E}^{\mathbb{P}_\epsilon}[(\phi^{-1}f)(X_T^\epsilon)]$ is differentiable at $\epsilon = 0$ and

$$\frac{1}{T}\,\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \mathbb{E}^{\mathbb{P}_\epsilon}[(\phi^{-1}f)(X_T^\epsilon)] \to 0\ .$$

Sensitivity Analysis

The vega

One way:
The method of Fournie et. al. with bounded-derivative coefficients

Fournie et. al.: Applications of Mallivin calculus to Monte Carlo methods in finance. inance Stoch. **3,** 391-412 (1999)

The perturbed process X_t^ϵ :

$$dX_t^\epsilon = b(X_t^\epsilon)\,dt + (\sigma + \epsilon\overline{\sigma})(X_t^\epsilon)\,dW_t$$

The $\mathbb{P}_\epsilon-$dynamics of X_t^ϵ are

$$dX_t^\epsilon = (b + (\sigma + \epsilon\overline{\sigma})\varphi_\epsilon)(X_t^\epsilon)\,dt + (\sigma + \epsilon\overline{\sigma})(X_t^\epsilon)\,dB_t^\epsilon$$

Sensitivity Analysis

The vega

We take apart two perturbations by the chain rule.

$$dX_t^\rho = (b + (\sigma + \rho\overline{\sigma})\varphi_\rho)(X_t^\rho)\,dt + \sigma(X_t^\rho)\,dB_t^\rho\ ,$$
$$dX_t^\nu = (b + \sigma\varphi)(X_t^\nu)\,dt + (\sigma + \nu\overline{\sigma})(X_t^\nu)\,dB_t^\nu\ .$$

Then we have

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\mathbb{E}^{\mathbb{P}_\epsilon}[(\phi^{-1}f)(X_T^\epsilon)]$$
$$= \frac{\partial}{\partial\rho}\Big|_{\rho=0}\mathbb{E}^{\mathbb{P}_\rho}[(\phi^{-1}f)(X_T^\rho)] + \frac{\partial}{\partial\nu}\Big|_{\nu=0}\mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1}f)(X_T^\nu)]\ .$$

Sensitivity Analysis

Let Z_t be the variation process given by

$$dZ_t = (b + \sigma\varphi)'(X_t)Z_t\, dt + \overline{\sigma}(X_t)dB_t + \sum_{i=1}^{d} \sigma_i'(X_t)Z_t\, dB_{i,t}\ ,\ Z_0 = 0_d$$

where σ_i is the i-th column vector of σ and 0_d is the d-dimensional zero column vector.

Theorem

Suppose that $b + \sigma\varphi$ and $\phi^{-1}f$ are continuously differentiable with bounded derivatives. If $\frac{1}{T} \cdot \mathbb{E}^{\mathbb{P}}[|Z_T|] \to 0$ as $T \to \infty$, then

$$\lim_{T\to\infty} \frac{1}{T} \frac{\partial}{\partial\nu}\bigg|_{\nu=0} \mathbb{E}^{\mathbb{P}_\nu}[(\phi^{-1}f)(X_T^\nu)] = 0\ .$$

Sensitivity Analysis

The vega

One way : the Lamperti transform for univariate processes

The perturbed process is expressed by

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\, dt + \sigma_\epsilon(X_t^\epsilon)\, dW_t\,, \quad X_0^\epsilon = X_0 = \xi\,, \tag{3.2}$$

Define a function

$$u_\epsilon(x) := \int_\xi^x \sigma_\epsilon^{-1}(y)\, dy\,. \tag{3.3}$$

The vega

Then we have

$$du_\epsilon(X_t^\epsilon) = (\sigma_\epsilon^{-1} b_\epsilon - \frac{1}{2}\sigma_\epsilon')(X_t^\epsilon)\, dt + dW_t \ , \ u_\epsilon(X_0^\epsilon) = u_\epsilon(\xi) = 0 \ .$$

Set $U_t^\epsilon := u_\epsilon(X_t^\epsilon)$, then

$$dU_t^\epsilon = \delta_\epsilon(U_t^\epsilon)\, dt + dW_t \ , \ U_0^\epsilon = 0 \ , \tag{3.4}$$

where $\delta_\epsilon(\cdot) := \left(\left(\sigma_\epsilon^{-1} b_\epsilon - \frac{1}{2}\sigma_\epsilon'\right) \circ v_\epsilon\right)(\cdot)$.

Sensitivity Analysis

The delta

Let

$$p_T := \mathbb{E}^{\mathbb{Q}}_{\xi}[e^{-\int_0^T r(X_s)ds} f(X_T)]$$

The quantity of interest is for large T

$$\nabla_\xi \, p_T \; .$$

From the martingale extraction

$$p_T = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_s)ds} f(X_T)] = \phi(\xi)\, e^{-\lambda T} \cdot \mathbb{E}^{\mathbb{P}}[(\phi^{-1} f)(X_T)] \, ,$$

it follows that

$$\frac{\nabla_\xi \, p_T}{p_T} = \frac{\nabla_\xi \, \phi}{\phi(\xi)} + \frac{\nabla_\xi \, \mathbb{E}^{\mathbb{P}}_{\xi}[(\phi^{-1} f)(X_T)]}{\mathbb{E}^{\mathbb{P}}_{\xi}[(\phi^{-1} f)(X_T)]} \; .$$

Sensitivity Analysis

The long-term behavior of the delta

We have

$$\lim_{T\to\infty} \frac{\nabla_\xi\, p_T}{p_T} = \frac{\nabla_\xi\, \phi}{\phi(\xi)}$$

if

$$|\nabla_\xi\, \mathbb{E}_\xi^{\mathbb{P}}[(\phi^{-1} f)(X_T)]| \to 0\ .$$

Sensitivity Analysis

Let Y_t be the first variation process defined by

$$dY_t = (b + \sigma\varphi)'(X_t)Y_t\,dt + \sum_{i=1}^{d} \sigma_i'(X_t)Y_t\,dB_{i,t}\ ,\ \ Y_0 = I_d$$

where σ_i is the i-th column vector of σ and I_d is the $d \times d$ identity matrix.

Corollary

Assume that the functions $b + \sigma\varphi$ and σ are continuously differentiable with bounded derivatives. If $\mathbb{E}_\xi^{\mathbb{P}}(\phi^{-1}f)^2(X_T)$ and $\mathbb{E}_\xi^{\mathbb{P}}\|\sigma^{-1}(X_T)Y_T\|^2$ are bounded on $0 < T < \infty$, then $\mathbb{E}_\xi^{\mathbb{P}}[(\phi^{-1}f)(X_T)]$ is differentiable by ξ and $|\nabla_\xi\,\mathbb{E}_\xi^{\mathbb{P}}(\phi^{-1}f)(X_T)| \to 0$ as $T \to \infty$. Here, $\|\cdot\|$ is the matrix 2-norm.

Examples of Options

Examples of Options

The CIR short-interest rate model

Under a risk-neutral measure $\mathbb{Q}$, the interest rate r_t follows

$$dr_t = (\theta - ar_t)\,dt + \sigma\sqrt{r_t}\,dW_t$$

with $2\theta > \sigma^2$.

The short-interest rate option price

$$p_T := \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r_t\,dt} f(r_T)]$$

Want to know the behavior for large T of

$$\frac{\partial p_T}{\partial \theta},\ \frac{\partial p_T}{\partial a},\ \frac{\partial p_T}{\partial \sigma}$$

Examples

Assume $f(r)$ is a nonnegative continuous function on $r \in [0, \infty)$, which is not identically zero, and that the growth rate at infinity is equal to or less than e^{mr} with $m < \frac{a}{\sigma^2}$.

The associated second-order equation is

$$\mathcal{L}\phi(r) = \frac{1}{2}\sigma^2 r\phi''(r) + (\theta - ar)\phi'(r) - r\phi(r) = -\lambda\phi(r) \, .$$

Set $\kappa := \frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2}$. It can be shown that the martingale extraction with respect to

$$(\lambda, \phi(r)) := (\theta\kappa, e^{-\kappa r})$$

stabilizes f.

Examples

For large T, we have that

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\theta\kappa \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \theta} = -\frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial a} = \frac{\theta(\sqrt{a^2+2\sigma^2}-a)}{\sigma^2\sqrt{a^2+2\sigma^2}} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \sigma} = \frac{\theta(\sqrt{a^2+2\sigma^2}-a)^2}{\sigma^3\sqrt{a^2+2\sigma^2}} \,,$$

$$\lim_{T\to\infty} \frac{\partial}{\partial r_0} \ln p_T = -\frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2} \,.$$

Examples

The quadratic term structure model

Let X_t be a d-dimensional OU process under the risk-neutral measure $\mathbb{Q}$

$$dX_t = (b + BX_t)\,dt + \sigma\,dW_t$$

where b is a d-dimensional vector, B is a $d \times d$ matrix, and σ is a non-singular $d \times d$ matrix.

The short interest rate : $r(x) = \beta + \langle \alpha, x \rangle + \langle \Gamma x, x \rangle$ where the constant β, vector α and symmetric positive definite Γ are taken to be such that $r(x)$ is non-negative for all $x \in \mathbb{R}^d$.

Examples

Interested in:

$$p_T = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(X_t)\,dt} f(X_T)]$$

with suitably nice payoff function f.

Let V be the *stabilizing solution* (the eigenvalues of $B - 2aV$ have negative real parts) of

$$2VaV - B^\top V - VB - \Gamma = 0\,,$$

and let $u = (2Va - B^\top)^{-1}(2Vb + \alpha)$.

The martingale extraction stabilizes f is

$$(\lambda, \phi(x)) = (\beta - \frac{1}{2}u^\top au + tr(aV) + u^\top b\,,\ e^{-\langle u,x\rangle - \langle Vx,x\rangle})\,.$$

Examples

We have that for $1 \leq i, j \leq d$,

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\beta + \frac{1}{2} u^\top a u - tr(aV) - u^\top b \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial b_i} = \frac{\partial \lambda}{\partial b_i} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial B_{ij}} = \frac{\partial \lambda}{\partial B_{ij}} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \sigma_i} = \frac{\partial \lambda}{\partial \sigma_i} \,,$$

$$\lim_{T\to\infty} \frac{\nabla_\xi \, p_T}{p_T} = -u - 2V\xi \,.$$

Examples of Expected Utilities

Examples

The geometric Brownian motion

Assume that S_t satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

with $\mu - \frac{1}{2}\sigma^2 > 0$. Let $\mathbb{Q}$ be the objective measure. Interested in

$$p_T = \mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T^{\alpha}] \, .$$

The corresponding infinitesimal generator:

$$(\mathcal{L}\phi)(s) = \frac{1}{2}\sigma^2 s^2 \phi''(s) + \mu s \phi'(s) - r\phi(s) \, .$$

The stabilizing martingale extraction:

$$(\lambda, \phi(s)) := (r - \mu\alpha - \frac{1}{2}\sigma^2\alpha(\alpha - 1), s^{\alpha})$$

Examples

The geometric Brownian motion

With this (λ, ϕ), we conclude that

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -r + \mu\alpha + \frac{1}{2}\sigma^2\alpha(\alpha - 1) ,$$

$$\lim_{T\to\infty} \frac{1}{T} \frac{\partial}{\partial\mu} \ln p_T = -\frac{\partial\lambda}{\partial\mu} = \alpha ,$$

$$\lim_{T\to\infty} \frac{1}{T} \frac{\partial}{\partial\sigma} \ln p_T = -\frac{\partial\lambda}{\partial\sigma} = \sigma\alpha(\alpha - 1) ,$$

$$\lim_{T\to\infty} \frac{\partial}{\partial S_0} \ln p_T = \frac{\phi'(S_0)}{\phi(S_0)} = \frac{\alpha}{S_0} .$$

Examples

The Heston model

An asset X_t follows

$$dX_t = \mu X_t \, dt + \sqrt{v_t} X_t \, dZ_t \ ,$$
$$dv_t = (\gamma - \beta v_t) \, dt + \delta \sqrt{v_t} \, dW_t \ ,$$

where Z_t and W_t are two standard Brownian motions with $\langle Z, W \rangle_t = \rho t$ for the correlation $-1 \leq \rho \leq 1$.

Interested in:

$$p_T := \mathbb{E}^{\mathbb{Q}}[u(X_T)] = \mathbb{E}^{\mathbb{Q}}[X_T^{\alpha}]$$

Examples

The Heston model

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\mu} \ln p_T = \alpha$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\gamma} \ln p_T$$

$$= -\frac{1}{2}\alpha(1-\alpha) \cdot \frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2}$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\beta} \ln p_T = \frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

Examples

The Heston model

$$\lim_{T\to\infty}\frac{1}{T}\frac{\partial}{\partial\delta}\ln p_T = -\rho\alpha\cdot\frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}} + \frac{(\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta)^2}{\delta^3\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

$$\lim_{T\to\infty}\frac{1}{T}\frac{\partial}{\partial\rho}\ln p_T = -\frac{\alpha\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\alpha\beta+\rho\alpha^2\delta}{\delta\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

$$\lim_{T\to\infty}\frac{\partial}{\partial X_0}\ln p_T = \frac{\alpha}{X_0}$$

$$\lim_{T\to\infty}\frac{\partial}{\partial v_0}\ln p_T = -\frac{1}{2}\alpha(1-\alpha)\cdot\frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2}.$$

Examples of LEFTs

Examples of LEFTs

The sensitivity analysis of the expected utility and the return of an exchange-traded fund (ETF) is explored.

The leveraged ETF (LETF) L_t can be written by

$$\frac{L_t}{L_0} = \left(\frac{X_t}{X_0}\right)^{\beta} e^{-r(\beta-1)t-\frac{\beta(\beta-1)}{2}\int_0^t \sigma^2(X_u)/X_u^2\,du} \,.$$

We consider a power utility function of the form

$$u(c) = c^{\alpha}\,,\ 0 < \alpha \leq 1\,.$$

Interested in the sensitivity analysis of

$$p_T := \mathbb{E}^{\mathbb{Q}}[u(L_T)]$$

Examples of LEFTs

The 3/2 model

The dynamics of X_t follows the 3/2 model

$$dX_t = (\theta - aX_t)X_t\,dt + \sigma X_t^{3/2}\,dW_t$$

with $\theta, a, \sigma > 0$ under the objective measure $\mathbb{Q}$.

The expected utility and the return of LETF is

$$p_T := \mathbb{E}^{\mathbb{Q}}[u(L_T)] = \mathbb{E}^{\mathbb{Q}}[e^{-\frac{\alpha\beta(\beta-1)\sigma^2}{2}\int_0^T X_u\,du}\,X_T^{\alpha\beta}]\cdot e^{-r\alpha(\beta-1)T}\,.$$

Interested in the behavior for large T of

$$\frac{\partial p_T}{\partial \theta}\,,\ \frac{\partial p_T}{\partial a}\,,\ \frac{\partial p_T}{\partial \sigma}$$

Examples of LEFTs

The 3/2 model

The corresponding infinitesimal operator is

$$\frac{1}{2}\sigma^2 x^3 \phi''(x) + (\theta - ax)x\phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2 x\phi(x) = -\lambda\phi(x)\ .$$

Set

$$\ell := \sqrt{\left(\frac{1}{2}+\frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta-1)} - \left(\frac{1}{2}+\frac{a}{\sigma^2}\right)\ .$$

It can be shown that the martingale extraction with respect to

$$(\lambda, \phi(x)) := \left(\theta\ell\,,\ x^{-\ell}\right)$$

stabilizes $f(x) := x^{\alpha\beta}$ when $|\beta| \leq 3$

Examples of LEFTs

The 3/2 model

If $\frac{a}{\sigma^2} + 1 - \alpha\beta > 0$, then

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\theta \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)} - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right) \right) ,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \theta} = -\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)} - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right) ,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial a} = \frac{\theta \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)} - \left(\frac{\sigma}{2} + a\right) \right)}{\sigma^2 \sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)}} ,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \sigma} = \frac{2a\theta \left(\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)} - \left(\frac{1}{2} + \frac{a}{\sigma^2}\right) \right)}{\sigma^3 \sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta - 1)}} .$$

Conclusion

Thank you !