Robust hedging with tradable options under price impact

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Classical practice is not robust

Pricing under a selected model $\mathbb{P}$, physical measure

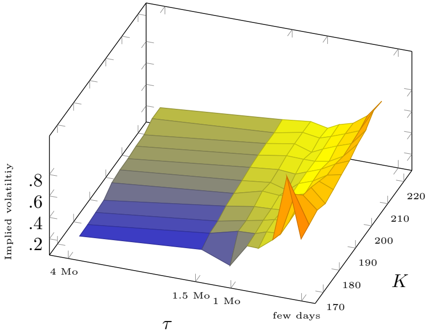
$$P := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi]$$

$\mathcal{M}$ = all martingale measures $\mathbb{Q}$ equivalent to $\mathbb{P}$.

Super-hedging price: $D := \inf\{a \ : \ a + (\Delta \cdot S)_T \geq \Phi \ \mathbb{P} - \text{a.s.}\}$.

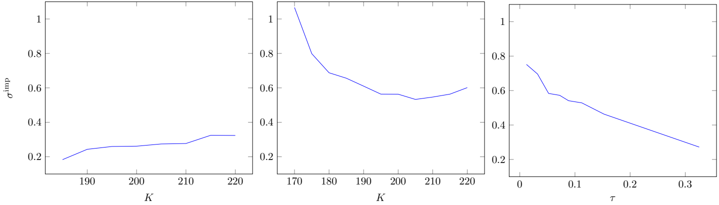
$$D = P$$

Volatility Surface: TSLA Feb 24, 2016



ATM=$175

Volatility smile: TSLA Feb 24, 2016



Implied volatility at Feb 24, 2016 for Tesla stock NASDAQ:TSLA. Left $\tau = 4$ Months, middle $\tau = 2$ weeks, and right $K = \$185$.

Durpire 96, Gyöngy 86, Krylov 85

Suppose all vanilla call prices are given by $C(K,T)$.
There is at most one Markov model which matches with the given call prices. In this case, the volatility is given by

$$\sigma^2(t,s) = 2\frac{C_T + rKC_K}{K^2 C_{KK}}\bigg|_{(T,K)=(t,s)}.$$

Issues and features

C_{KK} can be close to 0.
Volatility surface is not sufficiently smooth. Fitting a volatility surface can be done in many ways leading to different prices.
Markov (local vol) and hidden Markov (stochastic vol) models can be computationally feasible choices which can fit to the same volatility surface.

Two approaches to robust pricing

Robust pricing

(1) Model uncertainty
Class of eligible models $\mathcal{P}$: a collection of physical probability measures on some Polish space Ω

(2) Model-independent
Class of all pricing martingale measures $\mathcal{M}$ consistent with price quotes on the canonical space $\Omega = \mathbb{R}_+^T$
Asset price is the canonical process $S_t(S_1, \cdots, S_T) = S_t$

Semi-static Super-hedging

Semi-static Super-hedging

Class of eligible models $\mathcal{P}$

Asset price $S = (S_0, S_1, \cdots, S_T)$. Bond $B_t = 1$ for all t.

For $i \in I$ (finite set), ψ_i is a derivatives.

$$\Psi_{a,\eta,\Delta} := \underbrace{(\Delta \cdot S)_T}_{\text{Dynamic}} + a + \underbrace{\sum_{i \in I} \eta_i \psi_i}_{\text{Static}} \geq \Phi(S) \quad \mathcal{P} - \text{q.s.}$$

The cost of super-hedging portfolio is

$p(a, \eta, \Delta) := a + \sum_{i \in I} \eta_i p_i(\eta_i)$

$$\text{price of} \quad \psi_i : \quad p_i(\eta_i) = \begin{cases} a^i & \eta_i > 0 \\ b^i & \eta_i < 0 \end{cases}$$

$[b^i, a^i]$ is the bid-ask spread for ψ_i.

Arbitrage

Arbitrage

$p(a,\eta,\Delta) = 0$ and $\Psi_{a,\eta,\Delta} \geq 0$ $\mathcal{P}$-q.s., but $\mathbb{P}(\Psi_{a,\eta,\Delta} > 0) > 0$ for some $\mathbb{P} \in \mathcal{P}$.

$$\mathcal{M} := \{\mathbb{Q} \ : \ \mathbb{Q} << \mathbb{P} \ \text{ for some } \ \mathbb{P} \in \mathcal{P}, \\ S \text{ is a } \mathbb{Q} - \text{martingale and } b^i \leq \mathbb{E}^{\mathbb{Q}}[\psi_i] \leq a^i\}$$

Fundamental theorem (Bouchard-Nutz 13) (Bayraktar-Zhou 14)

NA $\equiv \forall \mathbb{P} \in \mathcal{P}, \exists \mathbb{Q} \in \mathcal{M}$ s.t. $\mathbb{P} << \mathbb{Q}$ (or $\mathcal{P}$ and $\mathcal{M}$ have the same polar sets).

Duality

Super-hedging (Bouchard-Nutz 13) (Bayraktar-Zhou 14)

If Φ is upper semi-analytic and NA holds:

$$D(\Phi) := \inf\{p(a,\eta,\Delta) \ : \ \Psi_{a,\eta,\Delta} \geq \Phi \quad \mathcal{P}-\text{q.s.}\} = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi]$$

Semi-static Super-hedging

$\Omega = \mathbb{R}_+^T$ is the canonical space. I can be infinite.

Semi-static Super-hedging

$$\Psi_{a,u,\Delta} := \underbrace{(\Delta \cdot S)_T}_{\text{Dynamic}} + a + \underbrace{\sum_{i \in J \subseteq I \text{ is finite}} \eta_i \psi_i}_{\text{Static}} \geq \Phi(S) \ \forall S \in \Omega$$

The cost of making the above super-hedging portfolio is $p(a, \eta, \Delta) := a + \sum_{i \in J} \eta_i p_i(\eta_i)$, where

$$\text{price of} \quad \psi_i : \quad p_i(\eta_i) = \begin{cases} a^i & \eta_i > 0 \\ b^i & \eta_i < 0 \end{cases}$$

$[b^i, a^i]$ is the bid-ask spread for ψ_i.

Model-independent arbitrage

Arbitrage

$p(a, \eta, \Delta) = 0$ and $\Psi_{a,\eta,\Delta} > 0$ for some $S \in \Omega$.

$$\mathcal{M} := \{\mathbb{Q} \ : \ S \text{ is a } \mathbb{Q} - \text{martingale and } b^i \leq \mathbb{E}^{\mathbb{Q}}[\psi_i] \leq a^i\}$$

Fundamental theorem (Acciaio et al. 2013)

NA $\equiv \mathcal{M} \neq \emptyset$.

Duality

Super-hedging (Acciaio et al. 2013)

If Φ is continuous with linear growth, there exists a super-linear option among ψ_is and NA holds:

$$D(\Phi) := \inf\{p(a,\eta,\Delta) \ : \ \Psi_{a,u,\Delta} \geq \Phi \quad \forall S \in \Omega\} = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi]$$

Liquid call options

$\forall K > 0$, $t = 1, \cdots, T$, $C(t, K)$ = price of call option which pays $(S_t - K)_+$ at time t. (Bid-Ask price equal)

(1) $K \mapsto C(t, K)$ is nonnegative and convex,

(2) $\lim_{K \downarrow 0} \partial_K C(t, K) \geq -1$,

(3) $\lim_{K \to \infty} C(t, K) = 0$

Marginals

$C(t, K) \to \mu_t$: $C(t, K) = \int (x - K)_+ \mu_t(dx)$

All feasible pricing measures:

$$\Pi := \{\mathbb{Q} \ : \ \mathbb{Q}|_t \sim \mu_t, \ \forall t\} \neq \emptyset$$

Semi-static Super-hedging

$u = (u_1, \cdots, u_T)$: u_t linear combination of call options maturing at t

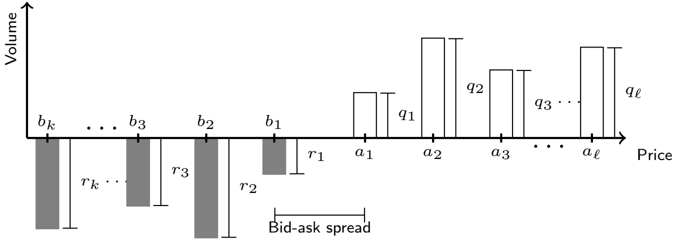
Semi-static Super-hedging

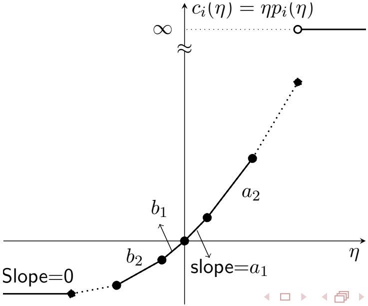
$$\Psi_{a,u,\eta,\Delta} := \underbrace{(\Delta \cdot S)_T}_{\text{Dynamic}} + a$$

$$+ \underbrace{\sum_t u_t + \sum_{i \in J \text{ is finite}} \eta_i \psi_i}_{\text{Static}} \geq \Phi(S), \quad \forall S$$

$$p(a,u,\eta) := a + \sum_t \int u_t d\mu_t + \sum_j \eta_j p_j(\eta_j)$$

Bid-ask chart for ψ_i





Portfolio constraint

Constraint $\Delta \in \mathcal{S}$

(i) $0 \in \mathcal{S}$.

(ii) For any $\Delta, \Delta' \in \mathcal{S}$ and any adapted process h with $h_t \in [0,1]$ for all $t = 0, \cdots, T-1$,

$$\{h_t \Delta_t + (1-h_t)\Delta'_t\}_{t=0}^{T-1} \in \mathcal{S}.$$

(iii) For any bdd $\Delta \in \mathcal{S}$, $\mathbb{Q} \in \Pi$, and $\varepsilon > 0$, there exist a closed set $D_\varepsilon \subseteq \mathbb{R}_+^T$ and a continuous $\Delta^\varepsilon \in \mathcal{S}$ such that

$$\mathbb{Q}(D_\varepsilon) > 1 - \varepsilon \ \text{ and } \ \Delta_t = \Delta_t^\varepsilon \text{ on } D_\varepsilon \text{ for } t = 0, \cdots, T-1.$$

Duality

Super-hedging price

$$D(\Phi) := \inf\{p(a,u,\eta) \ : \ \Delta \in \mathcal{S},\ \Psi_{a,u,\eta,\Delta} \geq \Phi \quad \forall S \in \Omega\}$$

Duality when $I = \emptyset$ and no constraint of Δ (Beiglböck et al. 2013)

If Φ u.s.c. continuous with linear growth, and $\mathcal{M} = \{\mathbb{Q} \in \Pi \ : \ S \text{ is a } \mathbb{Q}\text{-martingale}\}$

$$D(\Phi) = P(\Phi) := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi]$$

Monge-Kantorovich theory of optimal transportation

Optimal transportation duality

If Φ u.s.c. continuous with linear growth

$$\inf \left\{ \sum_t \int u_t d\mu_t \ : \ \sum_t u_t \geq \Phi \quad \forall S \in \Omega \right\} = \sup_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}}[\Phi]$$

Constraint and illiquid option

Penalty terms

$\{A_t^{\mathbb{Q}}\}_t$ (upper variation process for constraint $\mathcal{S}$) :

$$A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}} = \operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta \in \mathcal{S}} \Delta_t \mathbb{E}^{\mathbb{Q}}[S_{t+1} - S_t \mid \mathcal{F}_t]$$

Then, for any $\Delta \in \mathcal{S}$, $(\Delta \cdot S)_t - A_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-supermartingale and $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sup_{\Delta \in \mathcal{S}} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]$

Illiquid options with price impact

$$\mathcal{E}_I^{\mathbb{Q}} := \sup_{J \subseteq I(\text{finite})} \sup_{\eta} \sum_{j \in J} \eta_j (\mathbb{E}^{\mathbb{Q}}[\psi_j] - p_j(\eta_j))$$

Duality result

Duality (F-Huang 2014)

If Φ u.s.c. continuous with linear growth,

$$D_{\mathcal{S}} = \sup_{\mathbb{Q}\in\Pi} \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}} =: P_{\mathcal{S}}$$

No arbitrage

Model-independent arbitrage

$$p(a,u,\eta) = 0 \text{ and } \Psi_{a,u,\eta,\Delta} > 0 \quad \forall S \in \Omega$$

$$\mathcal{P}_{I,\mathcal{S}} := \{\mathbb{Q} \in \Pi \mid \{(\Delta \cdot S)_t\} \quad \text{is a } \mathbb{Q}\text{-supermartingale} \quad \forall \Delta \in \mathcal{S} \text{ and } b^i \leq \mathbb{E}^{\mathbb{Q}}[\psi_i] \leq a^i\}$$

Fundamental theorem (F-Huang 14)

NA $\equiv \mathcal{P}_{I,\mathcal{S}} \neq \emptyset$.

Remark

$$\mathcal{M}_{I,\mathcal{S}} \subseteq \mathcal{P}_{I,\mathcal{S}} \subseteq \Pi$$

Not necessarily $\mathcal{M}_{I,\mathcal{S}} = \mathcal{P}_{I,\mathcal{S}}$.

$\mathcal{M} \subseteq \mathcal{P} \subseteq \Pi$

Remark

For all $\Delta \in \mathcal{S}$

(1) if $\mathbb{Q} \in \mathcal{Q}_{I,\mathcal{S}}$, $\{(\Delta \cdot S)_t - A_t^Q\}_t$ is a local $\mathbb{Q}$-supermartingale.

(2) if $\mathbb{Q} \in \mathcal{P}_{I,\mathcal{S}}$, $\{(\Delta \cdot S)_t\}_t$ is a local $\mathbb{Q}$-supermartingale.

(3) If $\Delta \in \mathcal{S}^\infty$, locality is removed.

(4) If $1, -1 \in \mathcal{S}_{I,\mathcal{S}}$, then $\mathcal{P}_{I,\mathcal{S}} = \mathcal{M}_{I,\mathcal{S}}$.

(5) If $\mathcal{S}_{I,\mathcal{S}}$ contains all bounded strategies, $\mathcal{M}_{I,\mathcal{S}} = \mathcal{P}_{I,\mathcal{S}}$.

No-unbounded profit condition

Unbounded profit

For all $z \in \mathbb{R}_+$, there exists a $\Delta \in \mathcal{S}$, u, and η with

$$p(a,u,\eta) = -z \text{ and } \Psi_{a,u,\eta,\Delta} > 0 \quad \forall x \in \Omega$$

$$\mathcal{Q} := \{\mathbb{Q} \in \Pi \ : \ \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] < \infty, \ \mathcal{E}_I^{\mathbb{Q}} < \infty\}, \qquad \mathcal{P} \subseteq \mathcal{Q}$$

Fundamental theorem (F-Huang 14)

No unbounded profit $\equiv \mathcal{Q}_{I,\mathcal{S}} \neq \emptyset$.
Remark: Imagine $\mathcal{P}_{I,\mathcal{S}} = \emptyset$ (There is an arbitrage), but $\mathcal{Q}_{I,\mathcal{S}} \neq \emptyset$. Then, $D = P > -\infty$ and arbitrage is limited by a certain amount of profit.

Examples

Short-selling constraint

$$\mathcal{P} = \mathcal{Q}$$

Relative-drawdown constraint

Δ_t is bounded by two functions of $\frac{S_t}{S_t^*}$.

$$\mathcal{Q} = \Pi$$

No constraint

$$\mathcal{M} = \mathcal{P} = \mathcal{Q}$$

Non-tradable asset

$\Delta \equiv 0$, $\mathcal{P} = \mathcal{Q} = \Pi$.

Non-convex constraints

Gamma constraint

$$\mathcal{S}_\Gamma := \{\{\Delta_t\}_{t=0}^{T-1} : |\Delta_t - \Delta_{t-1}| \leq \Gamma, \ \forall t = 0, \cdots, T-1\},$$

$\Delta_{-1} \equiv 0$ (or any other constant)

Constraint $\mathcal{S}$ with bounded strategies

(i) $0 \in \mathcal{S}$.

(ii) (Boundedness) $\forall \Delta \in \mathcal{S}$, Δ is bounded.

(iii) (Continuous approximation) $\Delta \in \mathcal{S}$ can be approximated by continuous strategies in $\mathcal{S}$ in the sense described before.

Superhedging duality for bounded constraints

Penalty term

$C^{\mathbb{Q}} = \sup_{\Delta \in \mathcal{S}} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]$

$$\mathcal{Q}'_{\mathcal{S}} := \{\mathbb{Q} \in \Pi \mid C^{\mathbb{Q}} < \infty\}$$

$C^{\mathbb{Q}}$ and $\mathcal{Q}'_{\mathcal{S}}$ are analogous to $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]$ and $\mathcal{Q}_{\mathcal{S}}$ for convex constraint.

Duality (F-Huang 14)

$\Phi(x)$ u.s.c. with linear growth

$$D_{\mathcal{S}}(\Phi) = P'_{\mathcal{S}}(\Phi) := \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[\Phi] - C^{\mathbb{Q}}$$

No arbitrage

Fundamental theorem (F-Huang 14)

NA $\equiv \mathcal{P} := \{\mathbb{Q} \in \mathcal{Q} \mid C^{\mathbb{Q}} = 0\} \neq \emptyset$.

Example: Gamma constraint $\mathcal{M} = \mathcal{P}$.

Convex risk measures

Definition

A mapping $\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a *convex risk measure* if the following conditions are satisfied for all $\Phi, \Phi' \in \mathcal{X}$:

- *Monotonicity:* If $\Phi \leq \Phi'$, then $\rho(\Phi) \geq \rho(\Phi')$.
- *Translation Invariance:* If $m \in \mathbb{R}$, then $\rho(\Phi + m) = \rho(\Phi) - m$.
- *Convexity:* If $0 \leq \lambda \leq 1$, then $\rho(\lambda\Phi + (1-\lambda)\Phi') \leq \lambda\rho(\Phi) + (1-\lambda)\rho(\Phi')$.

Superhedging as a risk measure

$$\rho_{\mathcal{S}}(\Phi) := D_{\mathcal{S}}(-\Phi)$$

$$\mathcal{X} := \{\Phi \mid \rho_{\mathcal{S}}(\Phi) < \infty\}$$

Convex risk measures

Theorem (F-Huang 14)

Suppose $\mathcal{Q}_{\mathcal{S}} \neq \emptyset$.

$$\rho_{\mathcal{S}}(\Phi) = \sup_{\mathbb{Q} \in \Pi} \left(\mathbb{E}^{\mathbb{Q}}[-\Phi] - \alpha^*(\mathbb{Q}) \right),$$

where the *penalty function* α^* is given by

$$\alpha^*(\mathbb{Q}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] + \mathcal{E}_I^{\mathbb{Q}} & \text{if } \mathbb{Q} \in \mathcal{Q}_{I,\mathcal{S}}, \\ \infty, & \text{otherwise.} \end{cases}$$

Moreover, for any $\alpha : \Pi \mapsto \mathbb{R} \cup \{\infty\}$ such that (29) holds (with α^* replaced by α), we have $\alpha^*(\mathbb{Q}) \leq \alpha(\mathbb{Q})$ for all $\mathbb{Q} \in \Pi$.

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> "Finance theory consists of a set of concepts that help you to organize your thinking about how to allocate resources over time and a set of quantitative models to help you evaluate alternatives, make decisions, and implement them."
>
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Thank you for you attention.