

Obstacle problems defined by stable Lévy processes

Camelia Pop

School of Mathematics, University of Minnesota

Joint work with Nicola Garofalo, Arshak Petrosyan, and Mariana Smit Vega Garcia

1st Eastern Conference on Mathematical Finance
Worcester Polytechnic Institute
March 19, 2016

Outline

Motivation

Optimal regularity of solutions

Regularity of the free boundary

Selected references

Motivation I

- Pricing of (perpetual) American options when the underlying asset price is a pure-jump Markov process.

Motivation I

- Pricing of (perpetual) American options when the underlying asset price is a pure-jump Markov process.
- The asset price $\{S(t)\}_{t \geq 0}$ is characterized by the infinitesimal generator:

$$Au(x) = \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y| \leq 1\}}) d\nu(y) + b(x) \cdot \nabla u(x),$$

where $d\nu(y)$ is a Lévy measure.

Motivation I

- Pricing of (perpetual) American options when the underlying asset price is a pure-jump Markov process.
- The asset price $\{S(t)\}_{t \geq 0}$ is characterized by the infinitesimal generator:

$$Au(x) = \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y| \leq 1\}}) d\nu(y) + b(x) \cdot \nabla u(x),$$

where $d\nu(y)$ is a Lévy measure.

- Consider a perpetual American option written on the underlying $\{S(t)\}_{t \geq 0}$ with payoff $\varphi(x)$.

Motivation II

- We assume that the perpetual American option prices is given by

$$u(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} \varphi(S(\tau)) \mid S(0) = x \right],$$

where we assume that the asset price process $\{S(t)\}_{t \geq 0}$ is specified under a suitable probability measure.

Motivation II

- We assume that the perpetual American option prices is given by

$$u(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} \varphi(S(\tau)) \mid S(0) = x \right],$$

where we assume that the asset price process $\{S(t)\}_{t \geq 0}$ is specified under a suitable probability measure.

- We expect $u(x)$ to solve the system of complementarity conditions,

$$\begin{aligned} u &\geq \varphi && \text{on } \mathbb{R}^n, \\ -Au + ru &= 0 && \text{on } \{u > \varphi\}, \\ -Au + ru &\geq 0 && \text{on } \{u = \varphi\}, \end{aligned}$$

Motivation II

- We assume that the perpetual American option prices is given by

$$u(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} \varphi(S(\tau)) \mid S(0) = x \right],$$

where we assume that the asset price process $\{S(t)\}_{t \geq 0}$ is specified under a suitable probability measure.

- We expect $u(x)$ to solve the system of complementarity conditions,

$$\begin{aligned} u &\geq \varphi && \text{on } \mathbb{R}^n, \\ -Au + ru &= 0 && \text{on } \{u > \varphi\}, \\ -Au + ru &\geq 0 && \text{on } \{u = \varphi\}, \end{aligned}$$

or, more compactly, we have that

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall x \in \mathbb{R}^n.$$

Main questions

For the stationary obstacle problem,

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall x \in \mathbb{R}^n,$$

the main questions that we want to understand are:

Main questions

For the stationary obstacle problem,

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall x \in \mathbb{R}^n,$$

the main questions that we want to understand are:

1. Optimal regularity of solutions;

Main questions

For the stationary obstacle problem,

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall x \in \mathbb{R}^n,$$

the main questions that we want to understand are:

1. Optimal regularity of solutions;
2. Regularity of the free boundary, that is, of the topological boundary of the contact set $\{u = \varphi\}$.

Main questions

For the stationary obstacle problem,

$$\min\{-Au(x) + ru(x), u(x) - \varphi(x)\} = 0, \quad \forall x \in \mathbb{R}^n,$$

the main questions that we want to understand are:

1. Optimal regularity of solutions;
2. Regularity of the free boundary, that is, of the topological boundary of the contact set $\{u = \varphi\}$.

We will present results about the previous two questions in the case when the nonlocal operator A is the **fractional Laplacian with drift**, that is,

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x), \quad \forall x \in \mathbb{R}^n,$$

where $s \in (0, 1)$.

Pure-jump models in mathematical finance

Pure-jump models in mathematical finance

Jump processes are used to model asset price processes because

- Asset prices do not move continuously (small jumps can occur over small time intervals);
- Asset prices have heavy tails, which are incompatible with a Gaussian model.

Pure-jump models in mathematical finance

Jump processes are used to model asset price processes because

- Asset prices do not move continuously (small jumps can occur over small time intervals);
- Asset prices have heavy tails, which are incompatible with a Gaussian model.

For this reason, processes which allow for discontinuous paths and heavy tails in their distributions have been proposed to model asset prices.

Pure-jump models in mathematical finance

Jump processes are used to model asset price processes because

- Asset prices do not move continuously (small jumps can occur over small time intervals);
- Asset prices have heavy tails, which are incompatible with a Gaussian model.

For this reason, processes which allow for discontinuous paths and heavy tails in their distributions have been proposed to model asset prices.

Relevant to our research are models for asset prices that can be written as a **subordinated Brownian motion**

Pure-jump models in mathematical finance

Jump processes are used to model asset price processes because

- Asset prices do not move continuously (small jumps can occur over small time intervals);
- Asset prices have heavy tails, which are incompatible with a Gaussian model.

For this reason, processes which allow for discontinuous paths and heavy tails in their distributions have been proposed to model asset prices.

Relevant to our research are models for asset prices that can be written as a **subordinated Brownian motion** such as

- Normal Inverse Gaussian processes (Barndorff-Nielsen (1997-1998));
- Variance Gamma processes (Madan and Seneta (1990));
- Tempered stable processes (Koponen (1995), Boyarchenko and Levendorskiĭ (2000), Carr, Geman, Madan, and Yor (2002-2003)).

Normal Inverse Gaussian process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.

Normal Inverse Gaussian process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be the subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{\sqrt{2\pi x^3/2}} e^{-x/2} \mathbf{1}_{\{x>0\}}.$$

Normal Inverse Gaussian process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be the subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{\sqrt{2\pi x^3/2}} e^{-x/2} \mathbf{1}_{\{x>0\}}.$$

- The process $X(t) := Z(T(t))$ is called a Normal Inverse Gaussian process and is characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|),$$

where $A = \theta$, $B = \sqrt{\theta^2 + 1}$, $C = B/(2\pi)$, and $K_1(z)$ is the modified Bessel function of the second kind.

Normal Inverse Gaussian process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be the subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{\sqrt{2\pi x^3/2}} e^{-x/2} \mathbf{1}_{\{x>0\}}.$$

- The process $X(t) := Z(T(t))$ is called a Normal Inverse Gaussian process and is characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|),$$

where $A = \theta$, $B = \sqrt{\theta^2 + 1}$, $C = B/(2\pi)$, and $K_1(z)$ is the modified Bessel function of the second kind.

- The infinitesimal generator of $X(t)$ is

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} (u(x+y) - u(x)) d\nu(y) \\ &= 1 - (-\Delta u - 2\theta \cdot \nabla u + 1)^{1/2}(x). \end{aligned}$$

Inverse Gaussian subordinator

- The subordinator of the Normal Inverse Gaussian process can be written as the **inverse local time** of a one-dimensional Brownian motion with drift, with infinitesimal generator,

$$Lu(y) = \frac{1}{2} \frac{d^2 u(y)}{dy^2} + \frac{du(y)}{dy}, \quad \forall y > 0.$$

Inverse Gaussian subordinator

- The subordinator of the Normal Inverse Gaussian process can be written as the **inverse local time** of a one-dimensional Brownian motion with drift, with infinitesimal generator,

$$Lu(y) = \frac{1}{2} \frac{d^2 u(y)}{dy^2} + \frac{du(y)}{dy}, \quad \forall y > 0.$$

- We can write L as a Sturm-Liouville operator in the form,

$$Lu(y) = \frac{1}{2m(y)} \frac{d}{dy} \left(m(y) \frac{du}{dy} \right) (y), \quad \forall y > 0,$$

where we used the weight function,

$$m(y) = 2e^{2y}.$$

Dirichlet-to-Neumann map

- This implies that the generator of the Variance Gamma process is the Dirichlet-to-Neumann map for the extension operator:

$$Ev(x, y) = \frac{1}{2}v_{xx} + \theta v_x + \frac{1}{2}v_{yy} + v_y,$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

Dirichlet-to-Neumann map

- This implies that the generator of the Variance Gamma process is the Dirichlet-to-Neumann map for the extension operator:

$$Ev(x, y) = \frac{1}{2}v_{xx} + \theta v_x + \frac{1}{2}v_{yy} + v_y,$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

- In other words, we have that if $v \in C(\mathbb{R} \times [0, \infty))$ is a solution to the Dirichlet problem,

$$\begin{cases} Ev(x, y) = 0, & \forall (x, y) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) = v_0(x), & \forall x \in \mathbb{R}, \end{cases}$$

then we have that

$$\lim_{y \downarrow 0} m(y)v_y(x, y) = 2 \lim_{y \downarrow 0} v_y(x, y) = Av_0(x), \quad \forall x \in \mathbb{R}.$$

Variance Gamma process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.

Variance Gamma process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be a subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{x} e^{-x} \mathbf{1}_{\{x>0\}}.$$

Variance Gamma process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be a subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{x} e^{-x} \mathbf{1}_{\{x>0\}}.$$

- The process $X(t) := Z(T(t))$ is called a Variance Gamma process and is characterized by the Lévy measure,

$$\nu(x) = \frac{1}{|x|} e^{Ax - B|x|},$$

where $A = \theta$ and $B = \sqrt{\theta^2 + 2}$.

Variance Gamma process

- Let $Z(t) = W(t) + \theta t$ be a Brownian motion with drift.
- Let $T(t)$ be a subordinator with Lévy measure given by

$$\rho(x) = \frac{1}{x} e^{-x} \mathbf{1}_{\{x>0\}}.$$

- The process $X(t) := Z(T(t))$ is called a Variance Gamma process and is characterized by the Lévy measure,

$$\nu(x) = \frac{1}{|x|} e^{Ax - B|x|},$$

where $A = \theta$ and $B = \sqrt{\theta^2 + 2}$.

- The infinitesimal generator of $X(t)$ is

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} (u(x+y) - u(x)) d\nu(y) \\ &= -\log \left(-\frac{1}{2} \Delta u - \theta \cdot \nabla u + 1 \right) (x). \end{aligned}$$

Gamma subordinator

- Donati-Martin and Yor (2005) prove that the subordinator of the Variance Gamma process can be written as the **inverse local time** of a one-dimensional diffusion process with infinitesimal generator,

$$Lu(y) = \frac{1}{2} \frac{d^2 u(y)}{dy^2} + \left(\frac{1}{2y} + \sqrt{2} \frac{K_0'(\sqrt{2}y)}{K_0(\sqrt{2}y)} \right) \frac{du(y)}{dy}, \quad \forall y > 0,$$

where K_0 is the modified Bessel function of the second kind.

Gamma subordinator

- Donati-Martin and Yor (2005) prove that the subordinator of the Variance Gamma process can be written as the **inverse local time** of a one-dimensional diffusion process with infinitesimal generator,

$$Lu(y) = \frac{1}{2} \frac{d^2 u(y)}{dy^2} + \left(\frac{1}{2y} + \sqrt{2} \frac{K_0'(\sqrt{2}y)}{K_0(\sqrt{2}y)} \right) \frac{du(y)}{dy}, \quad \forall y > 0,$$

where K_0 is the modified Bessel function of the second kind.

- We can write L as a Sturm-Liouville operator in the form,

$$Lu(y) = \frac{1}{2m(y)} \frac{d}{dy} \left(m(y) \frac{du}{dy} \right) (y), \quad \forall y > 0,$$

where we used the weight function,

$$m(y) = y \left(K_0(\sqrt{2}y) \right)^2.$$

Dirichlet-to-Neumann map

- This implies that the generator of the Variance Gamma process is the Dirichlet-to-Neumann map for the extension operator:

$$Ev(x, y) = \frac{1}{2}v_{xx} + \theta v_x + \frac{1}{2}v_{yy} + \left(\frac{1}{2y} + \sqrt{2} \frac{K'_0(\sqrt{2}y)}{K_0(\sqrt{2}y)} \right) v_y,$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

Dirichlet-to-Neumann map

- This implies that the generator of the Variance Gamma process is the Dirichlet-to-Neumann map for the extension operator:

$$Ev(x, y) = \frac{1}{2}v_{xx} + \theta v_x + \frac{1}{2}v_{yy} + \left(\frac{1}{2y} + \sqrt{2} \frac{K'_0(\sqrt{2}y)}{K_0(\sqrt{2}y)} \right) v_y,$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

- In other words, we have that if $v \in C(\mathbb{R} \times [0, \infty))$ is a solution to the Dirichlet problem,

$$\begin{cases} Ev(x, y) = 0, & \forall (x, y) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) = v_0(x), & \forall x \in \mathbb{R}, \end{cases}$$

then we have that

$$\lim_{y \downarrow 0} m(y)v_y(x, y) = Av_0(x), \quad \forall x \in \mathbb{R}.$$

Tempered stable processes

- A similar analysis can be done for the class of tempered stable processes, which are characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|^{1+\alpha}} e^{Ax - B|x|},$$

where A, B, C are positive constants, $A < B$, and $\alpha \in (0, 2)$.

Tempered stable processes

- A similar analysis can be done for the class of tempered stable processes, which are characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|^{1+\alpha}} e^{Ax - B|x|},$$

where A, B, C are positive constants, $A < B$, and $\alpha \in (0, 2)$.

- The Lévy measure of the subordinator corresponding to the tempered stable process is known in closed form.

Tempered stable processes

- A similar analysis can be done for the class of tempered stable processes, which are characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|^{1+\alpha}} e^{Ax - B|x|},$$

where A, B, C are positive constants, $A < B$, and $\alpha \in (0, 2)$.

- The Lévy measure of the subordinator corresponding to the tempered stable process is known in closed form.
- To our knowledge, it is not known a closed form expression for a one-dimensional diffusion whose inverse local time at the origin is equal to the subordinator of the tempered stable process.

Tempered stable processes

- A similar analysis can be done for the class of tempered stable processes, which are characterized by the Lévy measure,

$$\nu(x) = \frac{C}{|x|^{1+\alpha}} e^{Ax - B|x|},$$

where A, B, C are positive constants, $A < B$, and $\alpha \in (0, 2)$.

- The Lévy measure of the subordinator corresponding to the tempered stable process is known in closed form.
- To our knowledge, it is not known a closed form expression for a one-dimensional diffusion whose inverse local time at the origin is equal to the subordinator of the tempered stable process.
- Necessary and sufficient conditions for subordinators that can be written as inverse local time of generalized diffusions were obtained by Knight (1981), and Kotani and Watanabe (1982).

Obstacle problems for nonlocal operators

- Up to not long ago, viewing the nonlocal operator as a Dirichlet-to-Neumann map (or, equivalently, the underlying Lévy process as a subordinated Brownian motion, where the subordinator is the inverse local time of a one-dimensional diffusion) was the **unique** method to analyze obstacle problems for nonlocal operators.

Obstacle problems for nonlocal operators

- Up to not long ago, viewing the nonlocal operator as a Dirichlet-to-Neumann map (or, equivalently, the underlying Lévy process as a subordinated Brownian motion, where the subordinator is the inverse local time of a one-dimensional diffusion) was the **unique** method to analyze obstacle problems for nonlocal operators.
- Caffarelli, Ros-Oton and Serra (2016) develop a new method that applies to all homogeneous Lévy measures that are symmetric about the origin, and does not use the previous property.

Obstacle problems for nonlocal operators

- Up to not long ago, viewing the nonlocal operator as a Dirichlet-to-Neumann map (or, equivalently, the underlying Lévy process as a subordinated Brownian motion, where the subordinator is the inverse local time of a one-dimensional diffusion) was the **unique** method to analyze obstacle problems for nonlocal operators.
- Caffarelli, Ros-Oton and Serra (2016) develop a new method that applies to all homogeneous Lévy measures that are symmetric about the origin, and does not use the previous property.
- The above mentioned models used in mathematical finance do not in general satisfy the assumptions in the Caffarelli, Ros-Oton and Serra (2016) article.

Symmetric $2s$ -stable processes

- We use symmetric stable processes as models for more complex processes used in financial applications because they share many important properties with the previous mentioned processes.

Symmetric $2s$ -stable processes

- We use symmetric stable processes as models for more complex processes used in financial applications because they share many important properties with the previously mentioned processes.
- Symmetric $2s$ -stable processes are characterized by the Lévy measure

$$\nu(y) = \frac{1}{|y|^{n+2s}}, \quad \forall y \in \mathbb{R}^n.$$

Symmetric $2s$ -stable processes

- We use symmetric stable processes as models for more complex processes used in financial applications because they share many important properties with the previous mentioned processes.
- Symmetric $2s$ -stable processes are characterized by the Lévy measure

$$\nu(y) = \frac{1}{|y|^{n+2s}}, \quad \forall y \in \mathbb{R}^n.$$

- The generator of symmetric $2s$ -stable process can be represented in integral form as

$$Au(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y|<1\}}) \frac{1}{|y|^{n+2s}},$$

where $s \in (0, 1)$.

Symmetric $2s$ -stable processes

- We use symmetric stable processes as models for more complex processes used in financial applications because they share many important properties with the previous mentioned processes.
- Symmetric $2s$ -stable processes are characterized by the Lévy measure

$$\nu(y) = \frac{1}{|y|^{n+2s}}, \quad \forall y \in \mathbb{R}^n.$$

- The generator of symmetric $2s$ -stable process can be represented in integral form as

$$Au(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - y \cdot \nabla u(x) \mathbf{1}_{\{|y|<1\}}) \frac{1}{|y|^{n+2s}},$$

where $s \in (0, 1)$.

- Using a functional-analytic framework, we can also represent A as

$$Au = -(-\Delta)^s u.$$

Symmetric stable processes with drift

- We consider a generalization of symmetric stable processes by adding a drift component, that is, we study operators of the form

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \quad \forall x \in \mathbb{R}^n.$$

Symmetric stable processes with drift

- We consider a generalization of symmetric stable processes by adding a drift component, that is, we study operators of the form

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \quad \forall x \in \mathbb{R}^n.$$

- The strength of the gradient perturbation is most easily seen in the Fourier variables:

$$-Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^{2s} + ib(x) \cdot \xi + c(x)) \widehat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Symmetric stable processes with drift

- We consider a generalization of symmetric stable processes by adding a drift component, that is, we study operators of the form

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \quad \forall x \in \mathbb{R}^n.$$

- The strength of the gradient perturbation is most easily seen in the Fourier variables:

$$-Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^{2s} + ib(x) \cdot \xi + c(x)) \widehat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

- A can be viewed as a pseudo-differential operator with symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

Symmetric stable processes with drift

- We consider a generalization of symmetric stable processes by adding a drift component, that is, we study operators of the form

$$Au(x) = -(-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \quad \forall x \in \mathbb{R}^n.$$

- The strength of the gradient perturbation is most easily seen in the Fourier variables:

$$-Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^{2s} + ib(x) \cdot \xi + c(x)) \widehat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

- A can be viewed as a pseudo-differential operator with symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

- The properties of the symbol, $a(x, \xi)$, change depending on whether

$$2s < 1, \quad 2s = 1, \quad \text{or} \quad 2s > 1.$$

Properties of the symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

We have three cases:

$$2s < 1, \quad 2s = 1, \quad \text{or} \quad 2s > 1.$$

Properties of the symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

We have three cases:

$$2s < 1, \quad 2s = 1, \quad \text{or} \quad 2s > 1.$$

1. If $2s < 1$ (**supercritical regime**): the drift component in $a(x, \xi)$ has the strongest contribution and the operator is not elliptic, so standard theory does not apply.

Properties of the symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

We have three cases:

$$2s < 1, \quad 2s = 1, \quad \text{or} \quad 2s > 1.$$

1. If $2s < 1$ (**supercritical regime**): the drift component in $a(x, \xi)$ has the strongest contribution and the operator is not elliptic, so standard theory does not apply.
2. If $2s = 1$ (**critical regime**): the jump and drift component in $a(x, \xi)$ have the same contribution, but they imply different regularity properties.

Properties of the symbol

$$a(x, \xi) = |\xi|^{2s} + ib(x) \cdot \xi + c(x), \quad \forall x, \xi \in \mathbb{R}^n.$$

We have three cases:

$$2s < 1, \quad 2s = 1, \quad \text{or} \quad 2s > 1.$$

1. If $2s < 1$ (**supercritical regime**): the drift component in $a(x, \xi)$ has the strongest contribution and the operator is not elliptic, so standard theory does not apply.
2. If $2s = 1$ (**critical regime**): the jump and drift component in $a(x, \xi)$ have the same contribution, but they imply different regularity properties.
3. If $2s > 1$ (**subcritical regime**): the jump component in $a(x, \xi)$ has the strongest contribution, which makes the operator elliptic, and so we expect the standard properties of elliptic operators to hold.

Obstacle problem

When $2s > 1$, we study the **stationary obstacle problem** defined by the fractional Laplacian with drift,

$$\min\{-Au, u - \varphi\} = 0, \quad \text{on } \mathbb{R}^n,$$

Obstacle problem

When $2s > 1$, we study the **stationary obstacle problem** defined by the fractional Laplacian with drift,

$$\min\{-Au, u - \varphi\} = 0, \quad \text{on } \mathbb{R}^n,$$

and we prove:

- Existence, uniqueness, and optimal regularity C^{1+s} of solutions;

Obstacle problem

When $2s > 1$, we study the **stationary obstacle problem** defined by the fractional Laplacian with drift,

$$\min\{-Au, u - \varphi\} = 0, \quad \text{on } \mathbb{R}^n,$$

and we prove:

- Existence, uniqueness, and optimal regularity C^{1+s} of solutions;
- The $C^{1+\alpha}$ regularity of the regular part of the free boundary.

Optimal regularity of solutions

Existence and optimal regularity of solutions

Theorem (Existence and optimal regularity of solutions)

Let $1 < 2s < 2$.

Assume that $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$, and $c \in C^s(\mathbb{R}^n)$ is a nonnegative function. Assume that the obstacle function, $\varphi \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, is such that

$$(A\varphi)^+ \in L^\infty(\mathbb{R}^n).$$

Then there is a solution, $u \in C^{1+s}(\mathbb{R}^n)$, to the obstacle problem defined by the fractional Laplacian with drift.

Uniqueness of solutions

Theorem (Uniqueness of solutions)

Let $0 < 2s < 2$ and $\alpha \in ((2s - 1) \vee 0, 1)$.

Assume that $b \in C(\mathbb{R}^n; \mathbb{R}^n)$ is a Lipschitz continuous function, and $c \in C(\mathbb{R}^n)$ is such that there is a positive constant, c_0 , such that

$$c(x) \geq c_0 > 0, \quad \forall x \in \mathbb{R}^n.$$

Assume that the obstacle function, $\varphi \in C(\mathbb{R}^n)$.

Then there is a unique solution, $u \in C^{1+\alpha}(\mathbb{R}^n)$, to the obstacle problem defined by the fractional Laplacian with drift.

Stochastic representations of solutions

- Uniqueness of solutions is established by proving their stochastic representation.

Stochastic representations of solutions

- Uniqueness of solutions is established by proving their stochastic representation.
- Let $(\Omega, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and let $N(dt, dx)$ be a Poisson random measure with Lévy measure,

$$d\nu(x) = \frac{dx}{|x|^{n+2s}},$$

and let $\tilde{N}(dt, dx)$ be the compensated Poisson random measure.

Stochastic representations of solutions

- Uniqueness of solutions is established by proving their stochastic representation.
- Let $(\Omega, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and let $N(dt, dx)$ be a Poisson random measure with Lévy measure,

$$d\nu(x) = \frac{dx}{|x|^{n+2s}},$$

and let $\tilde{N}(dt, dx)$ be the compensated Poisson random measure.

- Let $\{X(t)\}_{t \geq 0}$ be the unique RCLL solution to the stochastic equation,

$$X(t) = X(0) + \int_0^t b(X(s-)) ds + \int_0^t \int_{\mathbb{R}^n \setminus \{0\}} x \tilde{N}(ds, dx), \quad \forall t > 0.$$

Stochastic representations of solutions

- Uniqueness of solutions is established by proving their stochastic representation.
- Let $(\Omega, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and let $N(dt, dx)$ be a Poisson random measure with Lévy measure,

$$d\nu(x) = \frac{dx}{|x|^{n+2s}},$$

and let $\tilde{N}(dt, dx)$ be the compensated Poisson random measure.

- Let $\{X(t)\}_{t \geq 0}$ be the unique RCLL solution to the stochastic equation,

$$X(t) = X(0) + \int_0^t b(X(s-)) ds + \int_0^t \int_{\mathbb{R}^n \setminus \{0\}} x \tilde{N}(ds, dx), \quad \forall t > 0.$$

- Then, if $u \in C^{1+\alpha}(\mathbb{R}^n)$ is a solution to the obstacle problem, for some $\alpha \in ((2s - 1) \vee 0, 1)$, we have that

$$u(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[e^{-\int_0^\tau c(X(s-)) ds} \varphi(X(\tau)) \right], \quad \forall x \in \mathbb{R}^n,$$

where \mathcal{T} denotes the set of stopping times.

Remarks on uniqueness

- The **Lipschitz continuity** of the vector field $b(x)$ is used to ensure the existence and uniqueness of solutions, $\{X(t)\}_{t \geq 0}$, to the stochastic equation.

Remarks on uniqueness

- The Lipschitz continuity of the vector field $b(x)$ is used to ensure the existence and uniqueness of solutions, $\{X(t)\}_{t \geq 0}$, to the stochastic equation.
- The condition that the zeroth order coefficient, $c(x) \geq c_0 > 0$, is used to ensure that the expression on the right-hand side of the stochastic representation is finite even for unbounded stopping times, τ .

Remarks on uniqueness

- The **Lipschitz continuity** of the vector field $b(x)$ is used to ensure the existence and uniqueness of solutions, $\{X(t)\}_{t \geq 0}$, to the stochastic equation.
- The condition that the zeroth order coefficient, $c(x) \geq c_0 > 0$, is used to ensure that the expression on the right-hand side of the stochastic representation is finite even for unbounded stopping times, τ .
- If $\{X(t)\}_{t \geq 0}$ were an asset price process, and the law of the process were a risk-neutral probability measure, then the stochastic representation indicates that u is the value function of an **perpetual American option** with payoff φ on the underlying $\{X(t)\}_{t \geq 0}$.

Optimal regularity of solutions

- The optimal regularity of solutions to the obstacle problem for the fractional Laplace operator **without** drift was studied by Caffarelli-Salsa-Silvestre (2008), under the assumption that the obstacle function, $\varphi \in C^{2,1}(\mathbb{R}^n)$, and by Silvestre (2007), under the assumption that the contact set $\{u = \varphi\}$ is convex.

Optimal regularity of solutions

- The optimal regularity of solutions to the obstacle problem for the fractional Laplace operator **without** drift was studied by Caffarelli-Salsa-Silvestre (2008), under the assumption that the obstacle function, $\varphi \in C^{2,1}(\mathbb{R}^n)$, and by Silvestre (2007), under the assumption that the contact set $\{u = \varphi\}$ is convex.
- To obtain the **optimal regularity** of solutions, we reduce our problem to an obstacle problem without drift,

$$\min\{(-\Delta)^s \tilde{u}, \tilde{u} - \tilde{\varphi}\} = 0 \quad \text{on } \mathbb{R}^n,$$

for which we can at most assume that $\tilde{\varphi} \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.

Optimal regularity of solutions

- The optimal regularity of solutions to the obstacle problem for the fractional Laplace operator **without** drift was studied by Caffarelli-Salsa-Silvestre (2008), under the assumption that the obstacle function, $\varphi \in C^{2,1}(\mathbb{R}^n)$, and by Silvestre (2007), under the assumption that the contact set $\{u = \varphi\}$ is convex.
- To obtain the **optimal regularity** of solutions, we reduce our problem to an obstacle problem without drift,

$$\min\{(-\Delta)^s \tilde{u}, \tilde{u} - \tilde{\varphi}\} = 0 \quad \text{on } \mathbb{R}^n,$$

for which we can at most assume that $\tilde{\varphi} \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.

- From now on we consider the reduced problem and we write u instead of \tilde{u} and φ instead of $\tilde{\varphi}$.

Extension operator

- For $s \in (0, 1)$, let $a = 1 - 2s$ and consider the degenerate-elliptic operator,

$$L_a v = \frac{1}{2} \Delta v + \frac{1 - 2s}{2y} \frac{\partial v}{\partial y},$$

Extension operator

- For $s \in (0, 1)$, let $a = 1 - 2s$ and consider the degenerate-elliptic operator,

$$L_a v = \frac{1}{2} \Delta v + \frac{1 - 2s}{2y} \frac{\partial v}{\partial y},$$

which can be written in divergence form as

$$L_a v(x, y) = \frac{1}{2m(y)} \operatorname{div}(m(y) \nabla v)(x, y), \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where $m(y) = y^a$.

Extension operator

- For $s \in (0, 1)$, let $a = 1 - 2s$ and consider the degenerate-elliptic operator,

$$L_a v = \frac{1}{2} \Delta v + \frac{1 - 2s}{2y} \frac{\partial v}{\partial y},$$

which can be written in divergence form as

$$L_a v(x, y) = \frac{1}{2m(y)} \operatorname{div}(m(y) \nabla v)(x, y), \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where $m(y) = y^a$.

- Molchanov-Ostrovskii (1969) and Caffarelli-Silvestre (2007) prove that, if v is a L_a -harmonic function such that

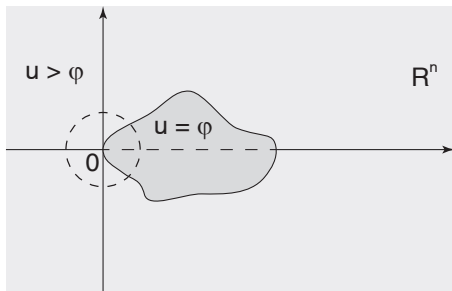
$$\begin{cases} L_a v(x, y) = 0, & \forall (x, y) \in \mathbb{R}^n \times (0, \infty), \\ v(x, 0) = v_0(x), & \forall x \in \mathbb{R}^n, \end{cases}$$

then we have that

$$\lim_{y \downarrow 0} m(y) v_y(x, y) = -(-\Delta)^s v_0(x), \quad \forall x \in \mathbb{R}^n.$$

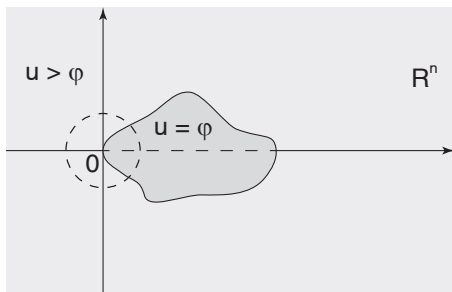
Steps to prove the optimal regularity of solutions

- We only need to study the regularity of the solutions in a neighborhood of free boundary points:



Steps to prove the optimal regularity of solutions

- We only need to study the regularity of the solutions in a neighborhood of free boundary points:



- We consider the height function

$$v(x) := u(x) - \varphi(x),$$

and the goal is to establish the [growth estimate](#):

$$0 \leq v(x) \leq C|x|^{1+s}.$$

Steps to prove the optimal regularity of solutions I

- Let $u(x, y)$ and $\varphi(x, y)$ be the L_a -harmonic extensions and let:

$$v(x, y) := u(x, y) - \varphi(x, y) + (-\Delta)^s \varphi(O) |y|^{1-a}, \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Steps to prove the optimal regularity of solutions I

- Let $u(x, y)$ and $\varphi(x, y)$ be the L_a -harmonic extensions and let:

$$v(x, y) := u(x, y) - \varphi(x, y) + (-\Delta)^s \varphi(O) |y|^{1-a}, \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

- Extend v by even symmetry across $\{y = 0\}$.

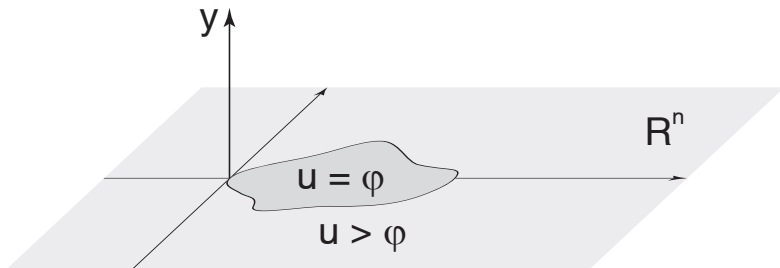
Steps to prove the optimal regularity of solutions I

- Let $u(x, y)$ and $\varphi(x, y)$ be the L_a -harmonic extensions and let:

$$v(x, y) := u(x, y) - \varphi(x, y) + (-\Delta)^s \varphi(O) |y|^{1-a}, \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

- Extend v by even symmetry across $\{y = 0\}$.
- The height function $v(x, y)$ satisfies the following conditions:

$$\begin{aligned} L_a v &= 0 && \text{on } \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}), \\ L_a v(x, y) &\leq h(x) \mathcal{H}^n|_{\{y=0\}} && \text{on } \mathbb{R}^{n+1}, \\ L_a v(x, y) &= h(x) \mathcal{H}^n|_{\{y=0\}} && \text{on } \mathbb{R}^{n+1} \setminus \{u = \varphi\}, \end{aligned}$$



Steps to prove the optimal regularity of solutions II

We need a suitable **monotonicity formula of Almgren-type** to find the lowest degree of regularity of the solution.

Steps to prove the optimal regularity of solutions II

We need a suitable **monotonicity formula of Almgren-type** to find the lowest degree of regularity of the solution.

Theorem (Almgren (1979))

Let u be a harmonic function. Then the function

$$\Phi_u(r) := r \frac{\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is non-decreasing in $r \in (0, 1)$.

Steps to prove the optimal regularity of solutions II

We need a suitable **monotonicity formula of Almgren-type** to find the lowest degree of regularity of the solution.

Theorem (Almgren (1979))

Let u be a harmonic function. Then the function

$$\Phi_u(r) := r \frac{\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is non-decreasing in $r \in (0, 1)$.

Moreover, $\Phi_u(r)$ is constant if and only if $\Phi_u(r) = k$, for some $k = 0, 1, 2, \dots$, and u is a homogeneous harmonic function of degree k .

Steps to prove the optimal regularity of solutions III

- We will establish a version of the monotonicity formula for the function:

$$\Phi_v^p(r) := r \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} |v|^2 |y|^{1-2s}, r^{n+1-2s+2(1+p)} \right\},$$

where r and p are positive constants.

Steps to prove the optimal regularity of solutions III

- We will establish a version of the monotonicity formula for the function:

$$\Phi_v^p(r) := r \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} |v|^2 |y|^{1-2s}, r^{n+1-2s+2(1+p)} \right\},$$

where r and p are positive constants.

- To see the connection with Almgren's classical monotonicity formula, omitting some technical details, the function $\Phi_v^p(r)$ takes the form:

$$\Phi_v^p(r) := 2r \frac{\int_{B_r} |\nabla v|^2 |y|^{1-2s}}{\int_{\partial B_r} v^2 |y|^{1-2s}} + (n+1-2s) + \text{"some noise"}.$$

Steps to prove the optimal regularity of solutions IV

Theorem (Almgren-type monotonicity formula)

Let $s \in (1/2, 1)$, $\alpha \in (1/2, s)$ and $p \in [s, \alpha + s - 1/2)$. Then there are positive constants, C and γ , such that the function

$$r \mapsto e^{Cr^\gamma} \Phi_v^p(r)$$

is non-decreasing, and we have that

$$\Phi_v(0+) \geq n + 1 - 2s + 2(1 + s).$$

Steps to prove the optimal regularity of solutions IV

Theorem (Almgren-type monotonicity formula)

Let $s \in (1/2, 1)$, $\alpha \in (1/2, s)$ and $p \in [s, \alpha + s - 1/2)$. Then there are positive constants, C and γ , such that the function

$$r \mapsto e^{Cr^\gamma} \Phi_v^p(r)$$

is non-decreasing, and we have that

$$\Phi_v(0+) \geq n + 1 - 2s + 2(1 + s).$$

Remark

Omitting some technical conditions, the lower bound

$$\Phi_v(0+) \geq n + 1 - 2s + 2(1 + s)$$

allows us to prove that the limit of the sequence of Almgren-type rescalings $\{v_r\}$, as $r \downarrow 0$, is a homogeneous function of degree at least $1 + s$.

Steps to prove the optimal regularity of solutions V

We study the properties of the sequence of **Almgren-type rescalings**:

$$v_r(x, y) := \frac{v(r(x, y))}{d_r}, \text{ where } d_r := \left(\frac{1}{r^{n+a}} \int_{\partial B_r} |v|^2 |y|^a \right)^{1/2}.$$

Steps to prove the optimal regularity of solutions V

We study the properties of the sequence of **Almgren-type rescalings**:

$$v_r(x, y) := \frac{v(r(x, y))}{d_r}, \text{ where } d_r := \left(\frac{1}{r^{n+a}} \int_{\partial B_r} |v|^2 |y|^a \right)^{1/2}.$$

Lemma (Uniform Schauder estimates)

Let $\alpha \in ((2s - 1) \vee 1/2, s)$ and $p \in [s, \alpha + s - 1/2)$.

Assume that $u \in C^{1+\alpha}(\mathbb{R}^n)$ and $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, and that $\liminf_{r \rightarrow 0} \frac{d_r}{r^{1+p}} = \infty$.

Then there are positive constants, $C, \gamma \in (0, 1)$ and r_0 , such that

$$\begin{aligned} \|v_r\|_{C^\gamma(\bar{B}_{1/8}^+)} &\leq C, \\ \|\partial_{x_i} v_r\|_{C^\gamma(\bar{B}_{1/8}^+)} &\leq C, \quad \forall i = 1, \dots, n, \\ \| |y|^a \partial_y v_r \|_{C^\gamma(\bar{B}_{1/8}^+)} &\leq C, \end{aligned}$$

for all $r \in (0, r_0)$.

Steps to prove the optimal regularity of solutions VI

- Almgren monotonicity formula and the compactness of the sequence of rescalings imply the growth estimate

$$0 \leq v(x) \leq C|x|^{1+s}, \quad \forall x \in B_{r_0}(O).$$

Steps to prove the optimal regularity of solutions VI

- Almgren monotonicity formula and the compactness of the sequence of rescalings imply the growth estimate

$$0 \leq v(x) \leq C|x|^{1+s}, \quad \forall x \in B_{r_0}(O).$$

- **Optimal regularity**, that is, $v \in C^{1+s}(\mathbb{R}^n)$, is a consequence of the preceding growth estimate of u .

Regularity of the free boundary

Classification of free boundary points

- The set of free boundary points: $\Gamma = \partial\{u = \varphi\}$.

Classification of free boundary points

- The set of free boundary points: $\Gamma = \partial\{u = \varphi\}$.
- Almgren's formula centered at $x_0 \in \Gamma$: $\Phi_{x_0}^p$.

Classification of free boundary points

- The set of free boundary points: $\Gamma = \partial\{u = \varphi\}$.
- Almgren's formula centered at $x_0 \in \Gamma$: $\Phi_{x_0}^p$.
- For all $p \in (s, 2s - 1/2)$ and for all $x_0 \in \Gamma$:

$$\Phi_{x_0}^p(0+) = n + a + 2(1 + s) \quad \text{or} \quad \Phi_{x_0}^p(0+) \geq n + a + 2(1 + p).$$

Classification of free boundary points

- The set of free boundary points: $\Gamma = \partial\{u = \varphi\}$.
- Almgren's formula centered at $x_0 \in \Gamma$: $\Phi_{x_0}^p$.
- For all $p \in (s, 2s - 1/2)$ and for all $x_0 \in \Gamma$:

$$\Phi_{x_0}^p(0+) = n + a + 2(1 + s) \quad \text{or} \quad \Phi_{x_0}^p(0+) \geq n + a + 2(1 + p).$$

- We define the set of **regular free boundary** points by

$$\Gamma_{1+s}(u) := \{x_0 \in \Gamma : \Phi_{x_0}^p(0+) = n + a + 2(1 + s)\}.$$

Classification of free boundary points

- The set of free boundary points: $\Gamma = \partial\{u = \varphi\}$.
- Almgren's formula centered at $x_0 \in \Gamma$: $\Phi_{x_0}^p$.
- For all $p \in (s, 2s - 1/2)$ and for all $x_0 \in \Gamma$:

$$\Phi_{x_0}^p(0+) = n + a + 2(1 + s) \quad \text{or} \quad \Phi_{x_0}^p(0+) \geq n + a + 2(1 + p).$$

- We define the set of **regular free boundary** points by

$$\Gamma_{1+s}(u) := \{x_0 \in \Gamma : \Phi_{x_0}^p(0+) = n + a + 2(1 + s)\}.$$

Theorem (Regularity of the regular free boundary)

The regular free boundary, $\Gamma_{1+s}(u)$, is a relatively open set and is locally $C^{1+\gamma}$, for a constant $\gamma = \gamma(n, s) \in (0, 1)$.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian **without** drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian **without** drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.
- Their method of the proof is based on monotonicity of the solution in a tangential cone of directions.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian **without** drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.
- Their method of the proof is based on monotonicity of the solution in a tangential cone of directions.
- This approach does not have an obvious generalization to the case when the obstacle function has a lower degree of monotonicity, that is, $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian **without** drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.
- Their method of the proof is based on monotonicity of the solution in a tangential cone of directions.
- This approach does not have an obvious generalization to the case when the obstacle function has a lower degree of monotonicity, that is, $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.
- Instead we adapt Weiss' approach (1998) of the proof of the regularity of the regular free boundary from the case of the Laplace operator to that of the fractional Laplacian, which in addition allows us to work with lower degree of regularity of the obstacle function.

Comparison with previous research

- The $C^{1+\gamma}$ regularity of the regular free boundary was obtained by Caffarelli-Salsa-Silvestre (2008) for the fractional Laplacian **without** drift in the case when the obstacle function $\varphi \in C^{2,1}(\mathbb{R}^n)$.
- Their method of the proof is based on monotonicity of the solution in a tangential cone of directions.
- This approach does not have an obvious generalization to the case when the obstacle function has a lower degree of monotonicity, that is, $\varphi \in C^{2s+\alpha}(\mathbb{R}^n)$, for all $\alpha \in (0, s)$.
- Instead we adapt Weiss' approach (1998) of the proof of the regularity of the regular free boundary from the case of the Laplace operator to that of the fractional Laplacian, which in addition allows us to work with lower degree of regularity of the obstacle function.
- A similar approach was employed in the study of the Signorini problem with variable coefficients by Garofalo-Petrosyan-Smit Vega Garcia (2015).

Main idea of the proof I

We fix a regular free boundary point $x_0 \in \Gamma_{1+s}$.

- Because we know the optimal regularity of solutions, we can now consider the **homogeneous rescalings**:

$$v_{x_0,r}(x,y) := \frac{1}{r^{1+s}} v(x_0 + rx, ry), \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}.$$

Main idea of the proof I

We fix a regular free boundary point $x_0 \in \Gamma_{1+s}$.

- Because we know the optimal regularity of solutions, we can now consider the **homogeneous rescalings**:

$$v_{x_0,r}(x,y) := \frac{1}{r^{1+s}} v(x_0 + rx, ry), \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}.$$

- The homogeneous rescalings converge to a non-trivial homogeneous solution in the class of functions:

$$\mathcal{H}_{1+s} := \left\{ a \left(x \cdot e + \sqrt{(x \cdot e)^2 + y^2} \right)^s \left(x \cdot e - s \sqrt{(x \cdot e)^2 + y^2} \right) : \right. \\ \left. a > 0, e \in \mathbb{R}^n, |e| = 1 \right\}.$$

Main idea of the proof II

For a regular free boundary point $x \in \Gamma_{1+s}$, let $|e_x| = 1$ and $a_x > 0$ be the defining parameters for the limit of the homogeneous rescalings at x .

Main idea of the proof II

For a regular free boundary point $x \in \Gamma_{1+s}$, let $|e_x| = 1$ and $a_x > 0$ be the defining parameters for the limit of the homogeneous rescalings at x .

Theorem (Garofalo-Petrosyan-P.-Smit (2015))

Let $x_0 \in \Gamma_{1+s}(u)$. Then there are positive constants C , η and $\gamma = \gamma(n, s)$, such that for all $x', x'' \in \Gamma \cap B_\eta(x_0)$, we have that

$$\begin{aligned} |a_{x'} - a_{x''}| &\leq C|x' - x''|^\gamma, \\ |e_{x'} - e_{x''}| &\leq C|x' - x''|^\gamma. \end{aligned}$$

Main idea of the proof II

For a regular free boundary point $x \in \Gamma_{1+s}$, let $|e_x| = 1$ and $a_x > 0$ be the defining parameters for the limit of the homogeneous rescalings at x .

Theorem (Garofalo-Petrosyan-P.-Smit (2015))

Let $x_0 \in \Gamma_{1+s}(u)$. Then there are positive constants C , η and $\gamma = \gamma(n, s)$, such that for all $x', x'' \in \Gamma \cap B_\eta(x_0)$, we have that

$$\begin{aligned} |a_{x'} - a_{x''}| &\leq C|x' - x''|^\gamma, \\ |e_{x'} - e_{x''}| &\leq C|x' - x''|^\gamma. \end{aligned}$$

- The $C^{1+\gamma}$ -regularity of the regular free boundary $\Gamma_{1+s}(u)$ is a direct consequence of the previous estimates.

Main idea of the proof II

For a regular free boundary point $x \in \Gamma_{1+s}$, let $|e_x| = 1$ and $a_x > 0$ be the defining parameters for the limit of the homogeneous rescalings at x .

Theorem (Garofalo-Petrosyan-P.-Smit (2015))

Let $x_0 \in \Gamma_{1+s}(u)$. Then there are positive constants C , η and $\gamma = \gamma(n, s)$, such that for all $x', x'' \in \Gamma \cap B_\eta(x_0)$, we have that

$$\begin{aligned} |a_{x'} - a_{x''}| &\leq C|x' - x''|^\gamma, \\ |e_{x'} - e_{x''}| &\leq C|x' - x''|^\gamma. \end{aligned}$$

- The $C^{1+\gamma}$ -regularity of the regular free boundary $\Gamma_{1+s}(u)$ is a direct consequence of the previous estimates.
- The previous estimates are a consequence of a version of a [Weiss monotonicity formula](#) and an [epiperimetric inequality](#) adapted to the framework of the fractional Laplacian.

Weiss-type monotonicity formula

- For $x_0 \in \Gamma_{1+s}$, we denote $v_{x_0}(x, y) := v(x_0 + x, y)$.

Weiss-type monotonicity formula

- For $x_0 \in \Gamma_{1+s}$, we denote $v_{x_0}(x, y) := v(x_0 + x, y)$.
- We define the **Weiss-type functional** by letting:

$$W_L(v, r, x_0) := \frac{1}{r^{n+2}} I_{x_0}(r) - \frac{1+s}{r^{n+3}} F_{x_0}(r), \quad \forall r \in \mathbb{R}_+,$$

$$I_{x_0}(r) := \int_{B_r(x_0)} |\nabla v_{x_0}|^2 |y|^a + \int_{B'_r(x_0)} v_{x_0} h_{x_0}, \quad \forall r \in \mathbb{R}_+,$$

$$F_{x_0}(r) := \int_{\partial B_r} |v_{x_0}(x_0 + \cdot)|^2 |y|^a,$$

where $B'_r = B_r \cap \{y = 0\}$.

Weiss-type monotonicity formula

- For $x_0 \in \Gamma_{1+s}$, we denote $v_{x_0}(x, y) := v(x_0 + x, y)$.
- We define the **Weiss-type functional** by letting:

$$W_L(v, r, x_0) := \frac{1}{r^{n+2}} I_{x_0}(r) - \frac{1+s}{r^{n+3}} F_{x_0}(r), \quad \forall r \in \mathbb{R}_+,$$

$$I_{x_0}(r) := \int_{B_r(x_0)} |\nabla v_{x_0}|^2 |y|^a + \int_{B'_r(x_0)} v_{x_0} h_{x_0}, \quad \forall r \in \mathbb{R}_+,$$

$$F_{x_0}(r) := \int_{\partial B_r} |v_{x_0}(x_0 + \cdot)|^2 |y|^a,$$

where $B'_r = B_r \cap \{y = 0\}$.

Theorem (Monotonicity of the Weiss functional)

There are constants $C, r_0 > 0$ such that for all $x_0 \in \Gamma(u)$ we have that

$$r \mapsto W_L(v, r, x_0) + Cr^{2s-1}$$

is nondecreasing on $(0, r_0)$.

Epiperimetric inequality

We define the **boundary adjusted Weiss energy** by letting:

$$W(v) := \int_{B_1} |\nabla v|^2 |y|^a - (1+s) \int_{\partial B_1} v^2 |y|^a.$$

Epiperimetric inequality

We define the **boundary adjusted Weiss energy** by letting:

$$W(v) := \int_{B_1} |\nabla v|^2 |y|^a - (1+s) \int_{\partial B_1} v^2 |y|^a.$$

Theorem (Epiperimetric inequality)

There are constants $\kappa, \delta \in (0, 1)$ such that if $w \in H^1(B_1, |y|^a)$ is a homogeneous function of degree $(1+s)$ such that

$$\begin{aligned} w &\geq 0 \quad \text{on } B_1 \cap \{y = 0\}, \\ \text{dist}(w, \mathcal{H}_{1+s}) &< \delta, \end{aligned}$$

then there is $\tilde{w} \in H^1(B_1, |y|^a)$ such that

$$\begin{aligned} \tilde{w} &\geq 0 \quad \text{on } B_1 \cap \{y = 0\}, \\ \tilde{w} &= w \quad \text{on } \partial B_1, \end{aligned}$$

and we have that $W(\tilde{w}) \leq (1 - \kappa)W(w)$.

Conclusions

Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.

Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.
- This allowed us to use local methods to adapt the concepts of monotonicity formulas already developed for model local operators to the framework of nonlocal operators.

Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.
- This allowed us to use local methods to adapt the concepts of monotonicity formulas already developed for model local operators to the framework of nonlocal operators.
- This is a property shared by many models important in financial engineering, such as the generators of the Variance Gamma process, Normal Inverse Gaussian process, and Tempered stable process.

Conclusions

- In the analysis of the obstacle problem for the fractional Laplacian with drift, it was essential to know that the fractional Laplacian operator can be viewed as the Dirichlet-to-Neumann map for a local extension operator.
- This allowed us to use local methods to adapt the concepts of monotonicity formulas already developed for model local operators to the framework of nonlocal operators.
- This is a property shared by many models important in financial engineering, such as the generators of the Variance Gamma process, Normal Inverse Gaussian process, and Tempered stable process.
- In the future, we hope to extend these methods to the study of the obstacle problem associated to the previously mentioned processes and their lower order perturbations.

THANK YOU!

Bibliography I



D. Applebaum

Lévy processes and stochastic calculus

Cambridge Studies in Advanced Mathematics (2004)



L. A. Caffarelli, X. Ros-Oton, J. Serra

Obstacle problems for integro-differential operators: Regularity of solutions and free boundaries

<http://arxiv.org/pdf/1601.05843.pdf> (2016)



L. A. Caffarelli, S. Salsa, L. Silvestre

Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian

Invent. Math. 171 (2008)



L. A. Caffarelli, L. Silvestre

An extension problem related to the fractional Laplacian

Comm. P.D.E. 32 (2007)

Bibliography II



P. Carr, H. Geman, D. B. Madan, M. Yor
Stochastic Volatility for Lévy Processes
Mathematical Finance 13 (2003)



R. Cont, P. Tankov
Financial modeling with jump processes
Chapman & Hall (2004)



C. L. Epstein, C. A. Pop
Regularity for the supercritical fractional Laplacian with drift
Journal of Geometric Analysis 26(2) (2016)
<http://arxiv.org/pdf/1309.5892.pdf>



A. Petrosyan, C. A. Pop
Optimal regularity of solutions to the obstacle problem for the fractional Laplacian with drift
Journal of Functional Analysis 268 (2015), no. 2
<http://arxiv.org/pdf/1403.5015.pdf>

Bibliography III



N. Garofalo, A. Petrosyan, C. A. Pop, M. Smit Vega Garcia

Regularity of the free boundary for the obstacle problem for the fractional Laplacian with drift

Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire, to appear (2016)

<http://arxiv.org/pdf/1509.06228.pdf>



M. E. Taylor

Partial Differential Equations I. Basic Theory

Springer (2011)



M. E. Taylor

Partial Differential Equations II. Qualitative studies of linear equations

Springer (2011)



L. Silvestre

Regularity of the obstacle problem for a fractional power of the Laplace operator

Comm. Pure Appl. Math. 60 (2007)