Systemic Risk Measures for Network Models

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Outline

Motivation

- **Goal of talk:**

  Novel approach to measurement of systemic risk

  - Systemic risk measures that are based on macroprudential objectives;
  - but enable at the same time capital regulation on the level of firms (i.e. obtain vectors of capital charges);

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  - Systemic risk measures that are based on macroprudential objectives;
  - but enable at the same time capital regulation on the level of firms (i.e. obtain vectors of capital charges);
  - provides a framework that is model independent;
  - methods for their implementation.
- Similar approaches taken in:
  - Biagini, Fouque, Frittelli & Meyer-Brandis (2015)
  - Armenti, Crepey, Drapeau & Papapantoleon (2015)

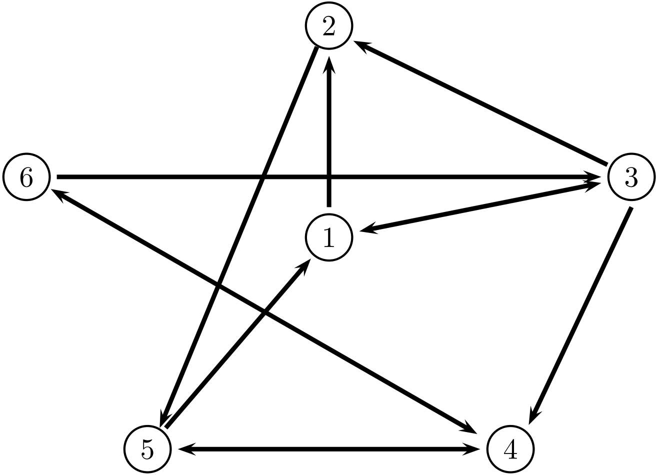
Models of Financial Contagion

Network Model with Local Interactions Only:
Eisenberg & Noe (2001)

- n financial firms
- Nominal liability matrix: $(\bar{p}_{ij})_{i,j=0,1,2,\ldots,n}$
- Total liabilities: $\bar{p}_i = \sum_{j=0}^{n} \bar{p}_{ij}$
- Relative liabilities:

$$a_{ij} = \begin{cases} \frac{\bar{p}_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0, \\ \frac{1}{n} & \text{if } \bar{p}_i = 0. \end{cases}$$

2.1 Models of Financial Contagion: Local Interactions



Network Model with Local Interactions Only:

- Liquid endowment: $x \in \mathbb{R}^n_+$
- Obligations fulfilled via transfers of the liquid asset.
- Equilibrium computed as fixed point: $p(x) \in \mathbb{R}^n_+$:

$$p_i(x) = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{n} a_{ji} p_j(x) \right), \quad i = 1, 2, ..., n$$

- Existence and Uniqueness results are well studied.

Network Model with Local and Global Interaction: Cifuentes, Shin & Ferrucci (2005), Amini, Filipovic & Minca (2015), Feinstein (2015), Feinstein & El-Masri (2015)

- Liquid endowment: $x \in \mathbb{R}^n_+$
- Obligations must be fulfilled via transfers of the liquid asset.
- Illiquid endowment in m assets: $S \in \mathbb{R}^{n \times m}_+$
- If necessary, illiquid positions must be liquidated, but these are subject to price impact described by the inverse demand function.
- Price impacts spread through the network via mark-to-market valuation.

Measures of Systemic Risk

The Basic Ingredients

1. **A notion of acceptability**
    - From a regulatory point of view, an **aggregate** of the financial sector is of primary importance.
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2. **An eligible "asset" in which supporting capital is invested**
   - In the case of a financial system, the **vector of capital endowments** is a suitable choice.
3. **A cash flow and random evolution model**
   - Cash flows are aggragated between financial institutions.
   - Can include cash flows **inside the financial system and to and from the real economy** modeled as well as the evolution of asset holdings.
   - Examples include models with local and global interaction

- **Financial firms:** $N = \{1, 2, \ldots, n\}$

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- **Assets** of financial firms evolve randomly between times $t = 0$ and $t = 1$ due to their primary business activities:
  - Sources of **risk**: $Y : \mathbb{R}^n \to L^\infty(\mathbb{R})$ *nondecreasing* random field
  - **Aggregation of the risk** by describing a statistic of interest for regulator
  - The random shocks are described by Y_k. Let $\mathcal{Y}$ be set of nondecreasing random fields.

Examples of aggregation:

- **Aggregation function without feedback:** $X \in L^\infty(\mathbb{R}^n)$ is future wealth values for each agent in the financial sector. Aggregate along cross-sectional profile by $\Lambda : \mathbb{R}^n \to \mathbb{R}$.

$$Y_k := \Lambda(X) + \sum_{i=1}^{n} k_i$$

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- **Aggregation function with feedback:**

$$Y_k := \Lambda(X + k)$$

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Special cases:

- **Conditional distribution:** Y_k^{-i} defined as the (aggregate) endowment of all firms $\{1, 2, ..., n\}$ conditional on stresses to firm i. This is the setting from $CoVaR$.
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- **Equity values and losses in a financial network:** The wealth of each agent evolves over time due to primary business activities. A secondary redistribution channel from contractual network, fire sales, etc. The impact on wealth to the economy outside of the financial sector is

$$Y_k := e_0(X + k)$$

Equity values and losses in a financial network:

- $X \in L^0(\mathbb{R}^n)$ and $S \in L^0(\mathbb{R}^m)$
- **Liquid assets** modeled by X
- **Parameters** by S (can include **illiquid** assets)
- **Updated portfolios** of agents at $t = 1$:

$$(k + X; S) = (k_1 + X_1, k_2 + X_2, \dots, k_n + X_n; S)$$

- The random shocks refer to the primary evolution of assets due to core business activities and systematic risk factors

- Secondary redistribution channel
  - **Contractual network** (e.g., loans or other securities) and **global interaction** (e.g., price impact, information)
  - **Outside economy** $i = 0$ is affected

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  - **Contractual network** (e.g., loans or other securities) and **global interaction** (e.g., price impact, information)
  - **Outside economy** $i = 0$ is affected
  - Increasing (in liquid assets) **E & L (equity and loss) function**

$$e : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+1} \cup \{-\infty\}$$

maps updated asset holdings $(k + X; S)$ to terminal wealth

$$e(k + X; S) = (e_0(k + X; S), e_1(k + X; S), \dots, e_n(k + X; S))$$

- $e_1(k + X; S), \dots, e_n(k + X; S)$ are equity of the firms, if positive, or losses caused by firms, if negative
- $e_0(k + X; S)$ terminal wealth of outside society

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- Vector of capital endowments: $k = (k_1, k_2, \ldots, k_n)$

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  $\longrightarrow$ **Eligible asset**
- Terminal statistic (e.g. wealth of real economy): Y_k

  $\longrightarrow$ **Cash flow and random evolution model**
- Acceptance set of classical scalar risk measure: $\mathcal{A}$
  - The regulator will require financial institutions to hold capital $k \in \mathbb{R}^n$ such that the resulting outcome is acceptable:

$$\mathbf{Y_k} \in \boldsymbol{\mathcal{A}}$$

  $\longrightarrow$ **Notion of acceptability**

Systemic Risk Measures

$R : \mathcal{Y} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n; \mathbb{R}^n_+) = \{D \subseteq \mathbb{R}^n \mid D = D + \mathbb{R}^n_+\}$ is a ***systemic risk measure*** if for some acceptance set $\mathcal{A} \subseteq L^\infty(\mathbb{R})$ of a scalar risk measure:

$$R(Y; m) = \{k \in \mathbb{R}^n \mid Y_{m+k} \in \mathcal{A}\}.$$

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Short: Write $R(Y)$ for $R(Y; 0)$.

Mild assumptions to exclude pathological cases:

- There exists $k \in \mathbb{R}^n$ so that $Y_k \in \mathcal{A}$.
- There exists $k \in \mathbb{R}^n$ so that $Y_k \notin \mathcal{A}$.

Such $Y \in \mathcal{Y}$ defined to be $\mathcal{L}$.

Systemic Risk Measures – Properties

1. Translativity:

$$R(Y; m+k) = R(Y; m) - k$$

2. Monotonicity:

$$Z \geq Y \;\Rightarrow\; R(Z) \supseteq R(Y)$$

Systemic Risk Measures – Properties

1. Translativity:

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2. Monotonicity:

$$Z \geq Y \;\Rightarrow\; R(Z) \supseteq R(Y)$$

3. Closed values: If $Y \in \mathcal{L}$ is continuous with respect to its indices in $\mathbb{R}^n$, then $R(Y)$ is a closed subset of $\mathbb{R}^n$.
4. Convexity: Suppose that $\mathcal{A}$ is convex. Can have different convexity properties depending on structure of random fields Y.

Examples:

- Aggregation without feedback: $Y_k = \Lambda(X) + \sum_{i=1}^n k_i$

$$R(Y) = \left\{k \in \mathbb{R}^n \mid \rho(\Lambda(X)) \leq \sum_{i=1}^n k_i\right\}$$

$$\rho(Z) := \inf\{u \in \mathbb{R} \mid Z + u \in \mathcal{A}\}$$

is all possible capital allocations for a framework in the sense of Chen, Iyengar, Moallemi (2013); Kromer, Overbeck, Zilch (2014).

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- Aggregation with feedback: $Y_k = \Lambda(X + k)$

$$R(Y) = \{k \in \mathbb{R}^n \mid X + k \in \mathcal{A}^e\}$$
$$\mathcal{A}^e := \{Z \in L^\infty(\mathbb{R}^n) \mid \Lambda(Z) \in \mathcal{A}\}$$

  is a set-valued risk measure in the sense of Meddeb, Jounini, Touzi (2004); Hamel, Heyde, Rudloff (2011).

Computation

- The determination of minimal capital requires the computation of the boundary of $R(Y)$. The algorithms checks if the condition

$$Y_k \in \mathcal{A} \tag{1}$$

is satisfied for points on a grid.
- Verifying condition (1) typically requires Monte Carlo simulation.

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- Verifying condition (1) typically requires Monte Carlo simulation.
- If aggregation without feedback then computation is simplified.
- A systemic risk measurement $R(Y)$ is an **upper set** that is also **convex** if Y is concave in its indices and $\mathcal{A}$ is convex.

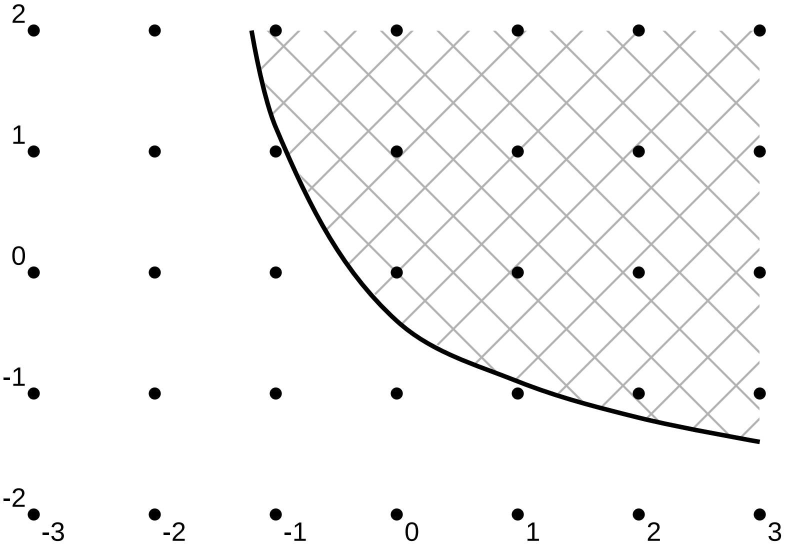
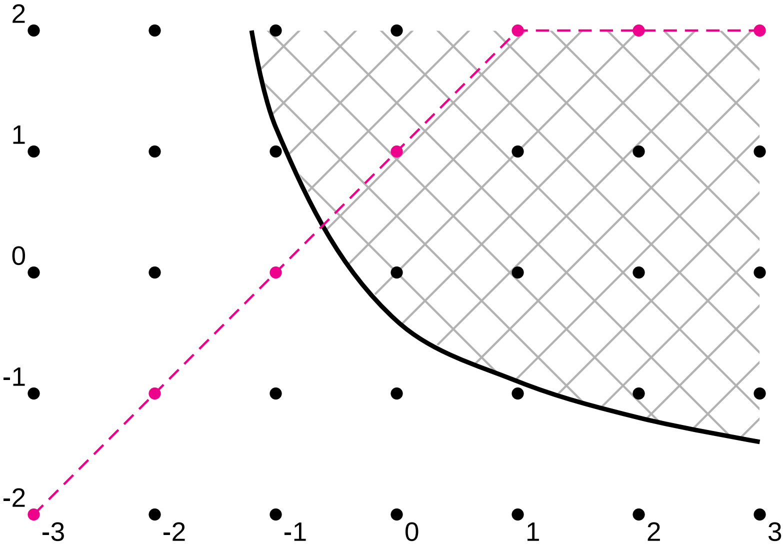


Figure: Grid search



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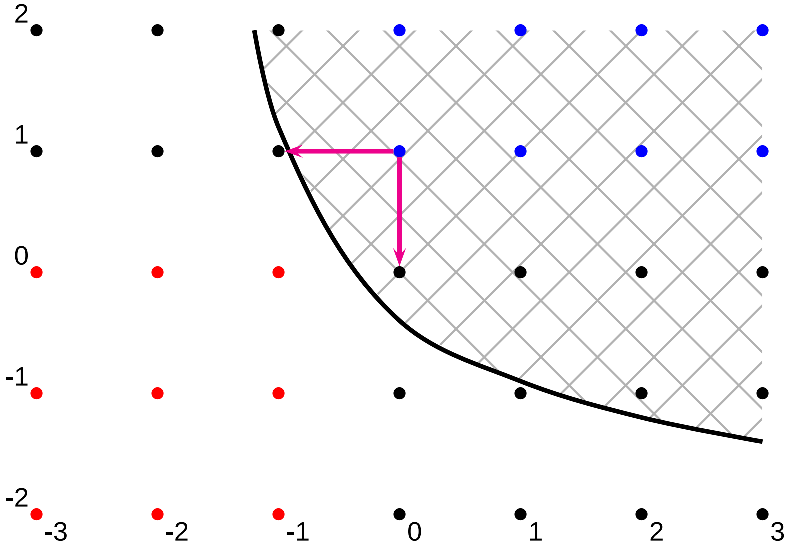

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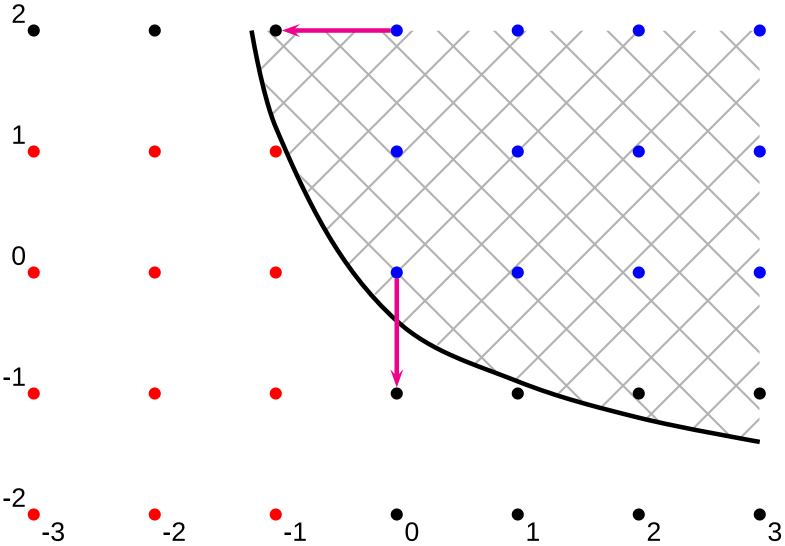

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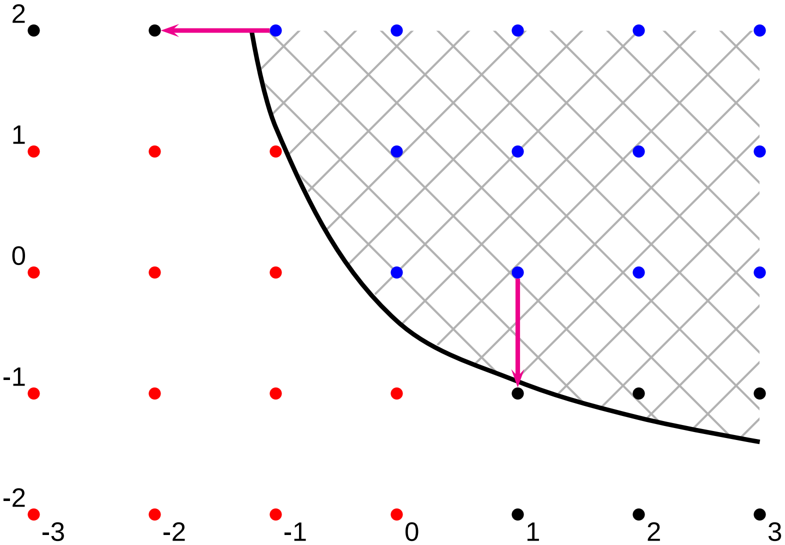

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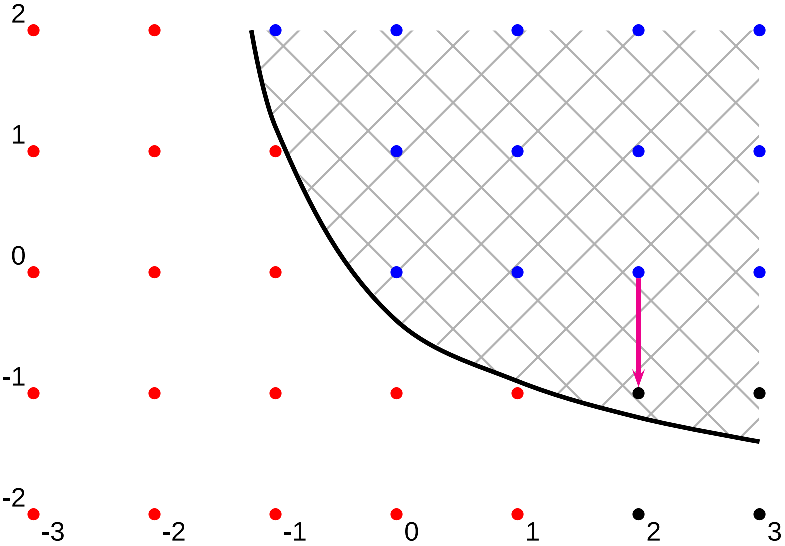

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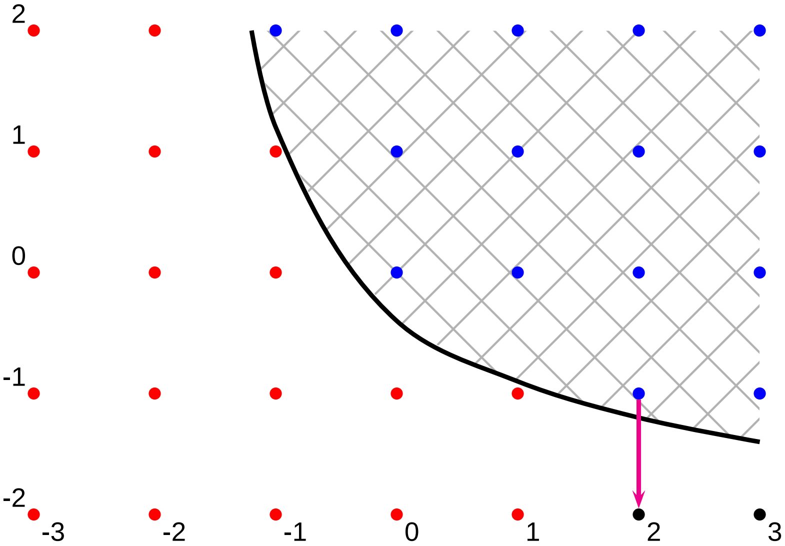

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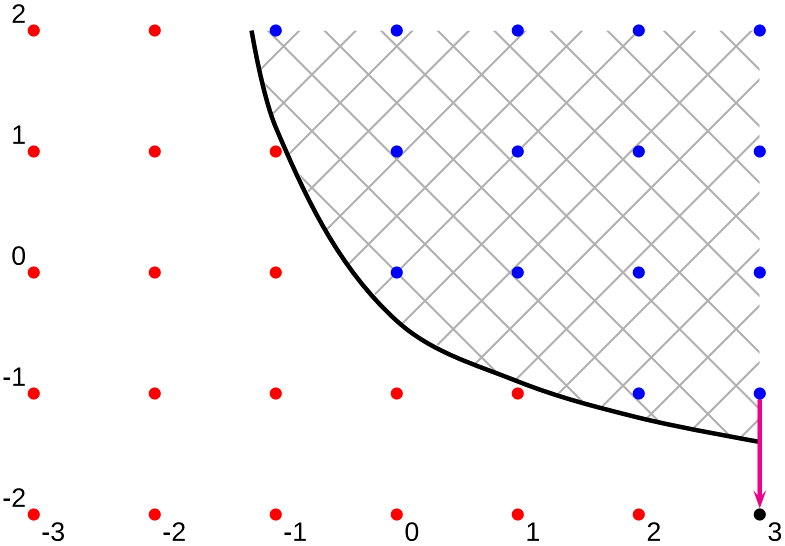

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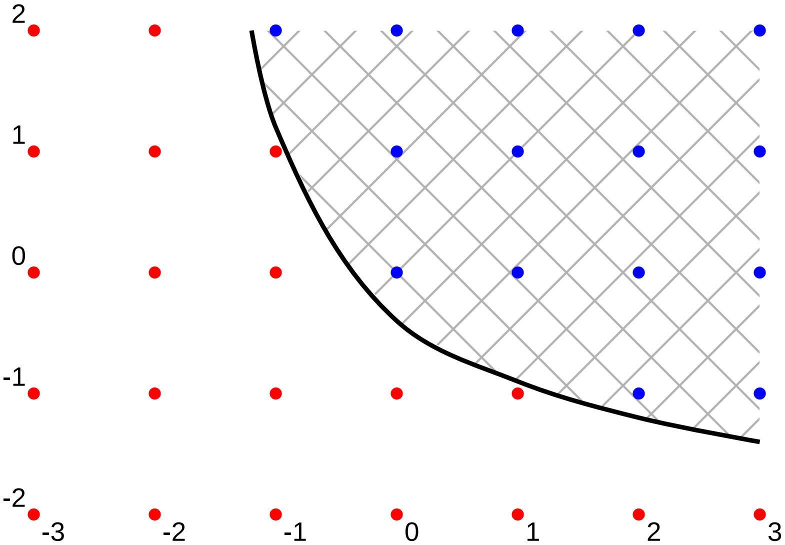

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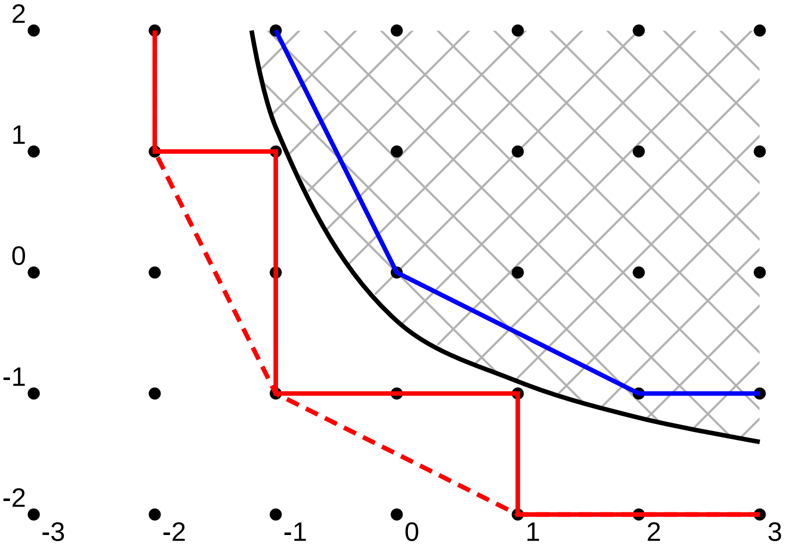

Figure: Grid search

Case studies

Case Study A

- Framework of Eisenberg & Noe (2001):
  only **local interaction** in the network
- Tiered graph:
  - Connections are randomly generated, probabilities within tiers and between tiers are fixed
  - Size of obligations within tiers and between tiers along connections fixed
- 2 Tiers/Groups: few firms with large obligations (10), many firms with small obligations (90)
- Further ingredients:
  random endowments, acceptance set defined by AV@R
- **Comparative statics: varying the degrees of connectedness**

Case Study A

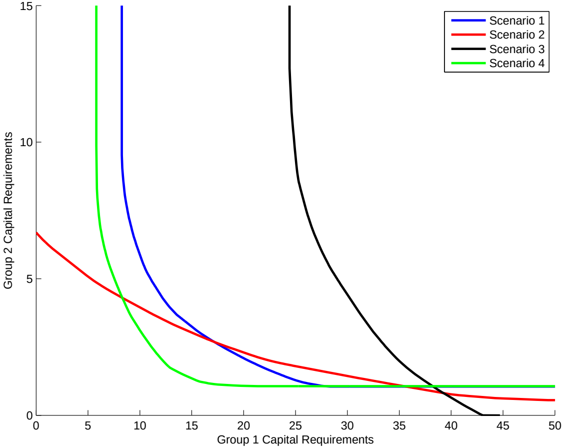

Figure: capital requirements for 2 tiers (100 banks; EISENBERG & NOE (2001) network model); AV@R

Case Study B

- Both **local (network) and global (price impact) interaction** (Amini, Filipovic, Minca (2013))
- Tiered graph:
  - Connections are randomly generated, probabilities within tiers and between tiers are fixed
  - Size of obligations within tiers and between tiers along connections fixed
- 3 Tiers/Groups: few large (10), intermediate number of intermediate size (90), many small (200)
- Further ingredients:
  random endowments, acceptance set defined by utility based shortfall risk measure with $u(x) = x^{\frac{3}{2}}$ and inverse demand function $f(x) = \exp(-x/2)$
- **Comparative statics: varying fraction in illiquid asset**

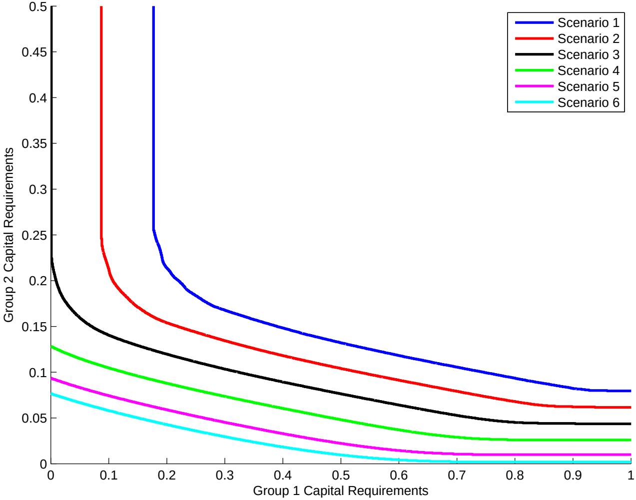

Figure: capital requirements for 3 tiers (300 banks; Amini, Filipovic, Minca (2013) network model); utility based shortfall risk measure

Orthant Risk Measures

Simplification

- $R(Y)$ is **collection** of acceptable capital allocations:
  - Financial firms cannot choose their capital independently of the other firms.
  - Although risk measurements $R(Y)$ capture the essence of systemic risk, they might be **difficult to communicate**.

Simplification

- $R(Y)$ is **collection** of acceptable capital allocations:
  - Financial firms cannot choose their capital independently of the other firms.
  - Although risk measurements $R(Y)$ capture the essence of systemic risk, they might be **difficult to communicate**.
- A simple alternative consists in choosing a point k^* in the boundary of $R(Y)$ and to require firms to hold capital inside

$$k^* + \mathbb{R}^n_+$$

  - Construction is more conservative, without any externalities of the choices of capital levels, and easy to communicate.

Orthant Risk Measures – Definition

Definition

Let $\mathcal{P}(\mathbb{R}^n)$ be the power set of $\mathbb{R}^n$. A mapping $k^* : \mathcal{L} \to \mathcal{P}(\mathbb{R}^n)$ is called an **orthant risk measure** *associated to a systemic risk measure R, if the following properties are satisfied:*

1. Minimal values: $k^*(Y) \subseteq \text{Min } R(Y), \quad Y \in \mathcal{L}$
2. Convex values: $\forall \alpha \in [0,1]$, $\forall Y \in \mathcal{L}$
$$k_1, k_2 \in k^*(Y) \ \Rightarrow \ \alpha k_1 + (1-\alpha)k_2 \in k^*(Y)$$
3. Translativity: $\forall Y \in \mathcal{L}$, $\forall k \in \mathbb{R}^n$
$$k^*(Y_{k+\cdot}) = k^*(Y) - k$$

Orthant Risk Measures – Characterization

Lemma

Let $R : \mathcal{L} \to \mathcal{P}(\mathbb{R}^n; \mathbb{R}^n_+)$ be a systemic risk measure. For $w : \mathcal{L} \to \mathbb{R}^n_+$ with $w(Y) \in (\mathrm{recc}(R(Y)))^+$ and $w(Y_{k+\cdot}) = w(Y)$, $k \in \mathbb{R}^n$, the set-valued mapping

$$\hat{k}(Y) = \operatorname{argmin}\left\{\sum_{i=1}^{n} w_i(Y)k_i \mid k \in R(Y)\right\} \tag{2}$$

defines an orthant risk measures. All orthant risk measures k^* are included in orthant risk measures $\hat{k}$ of form (2), i.e. $k^*(Y) \subseteq \hat{k}(Y)$ for all $Y \in \mathcal{L}$.

- The lemma provides examples of orthant risk measures via a specific choice of the "regulatory price of capital" w.

Options for the regulatory price of capital w:

Fixed price vectors:

- $w = (1, ..., 1)$: minimize system-wide addition of capital
- $w_i = 1/\bar{p}_i$: minimize total capital weighted by obligation
- w_i is a nonincreasing function of leverage:
  higher leverage $\Longrightarrow$ higher capital requirement

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Varying price vectors:

- w_i depends on the fluctuations of risk, e.g. $w_i = 1/var(X_i)$:
  higher variance of primary business activites $\Longrightarrow$ higher capital requirement

Some related literature:

- Brunnermeier & Cheridito (2013) is a special case. Generalization possible to include feedback effects
- Other recent scalar measures of systemic risk (not translative): Chen, Iyengar, Moallemi (2013), Kromer, Overbeck, Zilch (2014)
- Usually: scalar risk measure on aggregated losses; then risk attribution; no feedback effects

Thank You!

Feinstein, Rudloff, Weber (2016): Measures of systemic risk.