Multiresolution Path-planning with Traversal Costs based on Time-varying Spatial Fields

Raghvendra V. Cowlagi*  

Abstract—We address the problem of finding an optimal path for a vehicle in a planar environment where traversal costs are based on a time-varying spatial field defined over the environment. We assume that the accuracy with which the values of the spatial field are known gradually decreases with increasing distance from the vehicle’s current location. To extend previous results in the literature, we present a wavelet transform-based multiresolution motion-planning approach that incorporates traversal costs based on a time-varying spatial field. To enable fast online computations, we use 2D wavelet transforms with time-varying coefficients, instead of computing wavelet transforms with three independent variables. We discuss methods to compute time-varying traversal costs and an optimal path with a multiresolution approximation of the spatial field, we illustrate the proposed approach with a simulation example.

I. INTRODUCTION

Motion-planning for an unmanned aerial or terrestrial vehicle (UxV) is the problem of finding control inputs that enable the vehicle to travel from a prespecified initial position to a prespecified destination. The vehicle’s motion is described by a controlled dynamical system. The environment in which the vehicle moves may contain obstacles, and the vehicle trajectory is associated with a quality metric, i.e. a cost function. The objective of motion-planning is to find an obstacle-free vehicle trajectory of minimum cost. Motion-planning algorithms often involve discretization of the environment to first determine an obstacle-free path, which is a finite sequence of “waypoints” that approximately specifies the vehicle’s actual trajectory.

We consider a planar path-planning problem where the path cost depends on a time-varying scalar field defined over a planar workspace. We assume that the values taken by the scalar field —its intensity —are known at multiple resolutions. Specifically, the field intensity is accurately known in the immediate vicinity of the vehicle, and with progressively lesser accuracy in regions farther away.

There are several UxV applications that motivate path costs dependent on a time-varying scalar field. For example, UAVs will be required to incorporate the effects of adverse moving weather conditions (e.g. an advancing storm). A military UxV may be required to minimize the risk of detection by mobile adversaries. UxVs are expected to assist in emergency response to spatially escalating hazards such as a wildfire or the accidental/adversarial release of a toxic gas in the atmosphere. In these applications, a UxV may be required to deliver emergency supplies while minimizing its own exposure to the hazard, and/or disperse hazard retardants with maximal effect. The scalar field’s spatiotemporal evolution may be well-understood by physics-based models, and/or by real-time observations.

The motivations for considering the scalar field intensity at multiple resolutions is twofold. Firstly, in large environments, it may be necessary to consider multiresolution intensity data to ensure tractable computations with limited onboard memory and processing resources. To this end, high-resolution information about the scalar field’s intensity is more relevant in the vehicle’s immediate vicinity than in regions farther away. Secondly, uncertainty regarding the scalar field intensity may necessitate consideration of multiresolution intensity data. For example, the field intensity may be measured using a combination of local high-accuracy observations from the vehicle’s onboard sensors and wide-area low-accuracy observations from other sensors.

Related Work: Workspace cell decomposition is widely used in path-planning [1], [2]. The vehicle’s workspace is partitioned into convex regions called cells, that are either free or full of obstacles. A graph is defined such that each vertex of the graph uniquely corresponds to a free cell, and edges in this graph are defined according to geometric adjacency of cells. The vehicle’s path-planning is then transformed to a problem of path optimization in this graph, for which standard algorithms such as the Dijkstra, or the A* algorithm are available [3]. It is straightforward to include path costs based on a time-invariant field defined over the workspace. Optimal path-finding algorithms for graphs with time-varying edge costs appear in [4]–[7], but their applications to path-planning in a time-varying spatial field have not appeared.

Path-planning using multiresolution cell decompositions involves representing the vehicle’s environment with different levels of accuracy, e.g. the quadtree method [8]–[10]. Fast multiresolution cell decompositions, and associated path-planning techniques based on the discrete wavelet transform (DWT) have been discussed in [11]–[13]. The DWT is widely used in multiresolution signal processing, vision-based navigation, localization, and mapping [14]–[19]. In UxV applications, the wavelet transform can be a common standard of analysis of signals from multiple sensory sources [20]. Examples of applications of the DWT in multiresolution path-planning include [21]–[23].

We present a DWT-based multiresolution path-planning approach that incorporates traversal costs based on a time-varying spatial field. The proposed path planner shares algorithmic properties with that described in [13], which was proven to be complete. Instead of computing the

*Aerospace Engineering Program, Worcester Polytechnic Institute, Worcester, MA, USA. rvcowlagi@wpi.edu

1The property of completeness of a path-planning algorithm implies that the algorithm will find a path in a finite number of iterations if one exists, or will otherwise report failure in a finite number of iterations.
DWT with three (one temporal and two spatial) independent variables, we consider a 2D DWT of the spatial field with time-varying coefficients. We develop multiresolution path-planning based on these time-varying DWT coefficients. We present a simulation example to illustrate the proposed path-planning approach.

The contributions of this exploratory paper are as follows. We present a novel technique for path-planning with traversal costs based on a time-varying spatial field. The knowledge of such a spatial field involves large volumes of data. The proposed DWT-based representation of these data enables fast computations required for onboard real-time motion-planning. Furthermore, the proposed approach allows for the future development of motion control systems, where DWT coefficients can be used for signal analysis in navigation while simultaneously serving as a data structure for path-planning. We also develop calculations of path costs directly from the time-varying DWT-coefficients. These calculations are crucial for avoiding the slow approach of computing the DWT (or, for that matter, any other representation of the spatial field) with three independent variables.

We conclude the paper in Section V.

II. Problem Formulation

We consider a time-varying spatial field defined by the triplet \((R, \mathcal{F}, T)\), where \(R \subset \mathbb{R}^2\) is a compact, square region, \(T = [t_0, t_1] \subset \mathbb{R}_+\) is a compact interval, and \(\mathcal{F} : R \times T \rightarrow [0, 1]\) is a scalar intensity map that represents favorable regions in the environment with lower intensities. The field \(\mathcal{F}\) may represent, for instance, terrain elevation [21], a risk measure [11], a probabilistic occupancy grid [24], [25], or atmospheric concentration of a gas [26]. Without loss of generality, we assume that \(R = [0, 2^D] \times [0, 2^D]\), where \(D \in \mathbb{Z}_+\). Furthermore, we make the following assumptions.

Assumption 1. The field \(\mathcal{F}\) is sufficiently smooth such that

\[
\mathcal{F}(x,y, t_0) + \sum_{n=-\infty}^{\infty} \frac{(t-t_0)^n}{n!} \frac{\partial^n \mathcal{F}(x,y, t)}{\partial t^n} \bigg|_{t=t_0}
\]

converges to \(\mathcal{F}(x,y, t)\) for all \((x, y) \in R\), and \(t \in T\).

Assumption 2. The values taken by \(\mathcal{F}\) are known at a finite resolution \(m_t \geq D\), i.e. \(\mathcal{F}\) is piecewise constant over the square regions \(S_{m_t,k,l}\) (defined in Eqn. (A.3)), for \(k, l = 0, 1, \ldots, 2^D + m_t - 1\). Without loss of generality, \(m_t = 0\).

Assumption 3. The values taken by the temporal derivatives of \(\mathcal{F}\) at time \(t = t_0\), namely \(\frac{\partial^n \mathcal{F}(x,y, t)}{\partial t^n} \bigg|_{t=t_0}\) are known at the same finite resolution as described in Assumption 2.

Assumption 4. The vehicle moves at a constant speed of \(\delta t\)^{-1}, where \(\delta t > 0\) is prespecified.

Assumption 5. The vehicle is neither allowed to “wait” at any location in the environment, nor is it not allowed to move in cycles to emulate “waiting” while in constant motion.

Label-Correcting Algorithm with Time-varying Costs

procedure INITIALIZE\((\tilde{i}_S)\)
1: \(\mathcal{P} := \{\tilde{i}_S\}\), \(d(\tilde{i}_S) := 0\), \(\Theta(\tilde{i}_S) := t_0\)
2: for all \(j \in V \setminus \{\tilde{i}_S\}\) do
3: \(d(j) := \infty\)
procedure MAIN
1: INITIALIZE\((\tilde{i}_S)\)
2: while \(\mathcal{P} \neq \emptyset\) do
3: \(\tilde{i} := \text{REMOVE}(\mathcal{P})\)
4: for all \(j \in V\) such that \((\tilde{i}, j) \in E\) do
5: if \(d(\tilde{i}) + g(\tilde{i}, j, \Theta(\tilde{i})) < d(j)\) then
6: \(d(j) := d(\tilde{i}) + g(\tilde{i}, j, \Theta(\tilde{i}))\)
7: \(b(j) := \tilde{i}\), \(\Theta(j) := \Theta(\tilde{i}) + \delta t\)
8: \(\mathcal{P} := \text{INSERT}(\mathcal{P}, j)\)

Fig. 1. Pseudo-code for a label-correcting algorithm that solves Problem 1.

Assumption 1 is related to the smoothness of the temporal evolution of \(\mathcal{F}\). Fields arising from physical processes, such as diffusion, are expected to satisfy this assumption. Assumptions 2 and 3 recognize inherent numerical discretization: e.g. observations of \(\mathcal{F}\) and its derivatives may be available pointwise over a spatial grid. Assumptions 4 and 5 are for algorithmic simplicity.

Assumption 2 relates to workspace cell decomposition, in that each of the square regions \(S_{m_t,k,l}\) may be considered a cell. We denote by \(\Omega\) the set of all regions \(S_{m_t,k,l}\) in \(R\), for \(k, l \in \{0, 1, \ldots, 2^D - 1\}\). We construct a graph \(\tilde{G} = (\bar{V}, \bar{E})\) such that each region \(S_{m_t,k,l}\) is uniquely associated with a vertex in \(\bar{V}\), and each pair of regions with two common vertices is uniquely associated with an element in \(\bar{E}\). We denote by cell \((\tilde{i}; \Omega)\) the coordinates of the center of the cell associated with a vertex \(\tilde{i} \in \bar{V}\), and by \(\text{vert}(\tilde{c}; \tilde{G})\) the vertex in \(\bar{V}\) associated with a cell \(\tilde{c} \in \Omega\).

We first consider a baseline path-planning problem in the graph \(\tilde{G}\). Let \(\tilde{i}_S, \tilde{i}_G \in \bar{V}\) be prespecified initial and goal vertices. Following [13], a path \(\tilde{\pi}(\tilde{i}_S, \tilde{i}_G)\) in \(\tilde{G}\) is a finite sequence \((\tilde{i}_0, \ldots, \tilde{i}_P)\) of vertices with no repetition such that \(\{\tilde{i}_{k-1}, \tilde{i}_k\} \in \bar{E}\) for each \(k = 1, \ldots, P\), with \(\tilde{i}_0 = \tilde{i}_S\) and \(\tilde{i}_P = \tilde{i}_G\). The cost of a path \(\tilde{\pi}\) in \(\tilde{G}\) is:

\[
\tilde{J}(\tilde{\pi}) := \sum_{k=1}^{P} g(\tilde{i}_{k-1}, \tilde{i}_k, k\delta t), \quad \text{where} \quad (1)
\]

\[
g(\tilde{i}, j, t) := \frac{1}{2}\lambda_1 \left(\mathcal{F}(c_j, t) + \mathcal{F}(c_j, t + \frac{1}{2}\delta t)\right) + \lambda_2,
\]

and \(c_i := \text{cell}(\tilde{i}; \Omega)\), \(c_j := \text{cell}(\tilde{j}; \Omega)\), and \(\lambda_1, \lambda_2 \in (0, 1]\).

Problem 1. Find a path \(\tilde{\pi}^*\) in \(\tilde{G}\) such that for every other path \(\tilde{\pi}\) in \(\tilde{G}\), \(\tilde{J}(\tilde{\pi}^*) \leq \tilde{J}(\tilde{\pi})\).

By Assumption 5, every admissible path from \(\tilde{i}_S\) to \(\tilde{i}_G\) has a finite number of vertices. Therefore, there are a finite number of paths in \(\tilde{G}\), and a path with minimum cost exists.

A. Baseline Solution

Problem 1 can be solved by a variation of the standard label-correcting algorithm [3], as shown in Fig. 1. This algorithm searches for the optimal path from \(\tilde{i}_S\) to \(\tilde{i}_G\) by...
where

By Assumption 1, solving Problem 2 are provided next.

A. Vehicle-Centric Multiresolution Approximation

To [14], [15] for further details.

We consider the coefficients $t \mapsto a_{m_0,k,\ell}(t), d^p_{m,k,\ell}(t)$ defined analogously to Eqns. (A.1) and (A.2), as follows:

$$a_{m_0,k,\ell}(t) := \left\langle \Phi_{m_0,k,\ell}(x,y), F(x,y,t) \right\rangle,$$

$$d^p_{m,k,\ell}(t) := \left\langle \Psi^p_{m,k,\ell}(x,y), F(x,y,t) \right\rangle,$$

for $p = 1, 2, 3, k, \ell \in \mathbb{Z}, m_1 \geq m \geq m_0$, and $m_0 = -D$. By Assumption 1,

$$a_{m_0,k,\ell}(t) = a^0_{m_0,k,\ell} + \sum_{n=1}^{\infty} \frac{(t-t_0)^n}{n!} a^0_{m_0,k,\ell},$$

$$d^p_{m,k,\ell}(t) = d^p_{m,k,\ell} + \sum_{n=1}^{\infty} \frac{(t-t_0)^n}{n!} d^p_{m,k,\ell},$$

where $a^0_{m_0,k,\ell} := \left\langle \Phi_{m_0,k,\ell}(x,y), F(x,y,t_0) \right\rangle, d^p_{m_0,k,\ell} := \left\langle \Psi^p_{m_0,k,\ell}(x,y), F(x,y,t_0) \right\rangle,$ and

$$\alpha^0_{m_0,k,\ell} := \left\langle \Phi_{m_0,k,\ell}(x,y), \frac{\partial^n F(x,y,t)}{\partial t^n} \right\rangle_{t=t_0},$$

$$\beta^p_{m,k,\ell} := \left\langle \Psi^p_{m,k,\ell}(x,y), \frac{\partial^n F(x,y,t)}{\partial t^n} \right\rangle_{t=t_0}.$$
each vertex \(j \in V\) corresponds to a set \(W(j, V) \subseteq \tilde{V}\), and the collection \(\{W(j, V)\}_{j \in V}\) is a partition of \(\tilde{V}\). Specifically: \(W(j, V) := \{j \in V : \text{cell}(j; \Omega) \subseteq \text{cell}(j; \Omega_{\text{mr}})\}\). Two vertices \(i, j \in V\) are adjacent in \(\mathcal{G}\) if and only if there exist \(i \in W(i, V)\) and \(j \in W(j, V)\) such that \(\{i, j\} \in \tilde{E}\). By definition, edge costs \(g : E \to \mathbb{R}_+\) in the graph \(\mathcal{G}\) as

\[
g(i, j, t) := \frac{1}{2}(\lambda_1 f_i + \lambda_2 |W(i, V)|) + \frac{1}{2}(\lambda_1 f_j + \lambda_2 |W(j, V)|),
\]

where \(f_i := \int_{t + \Delta t_i}^{t + \Delta t_i + \Lambda} \bar{F}(\text{cell}(i; \Omega_{\text{mr}}), s) \, ds\), \(f_j := \int_{t + \Delta t_j}^{t + \Delta t_j + \Lambda} \bar{F}(\text{cell}(j; \Omega_{\text{mr}}), s) \, ds\), \(\Delta t_i = \frac{|W(i, V)|}{2\delta t}\), and \(\Delta t_j = \frac{|W(j, V)|}{2\delta t}\).

The cost \(J(\pi)\) of a path in the graph \(\mathcal{G}\) is the sum of the costs of all edges in the path. Note that the edge cost function defined in Eqs. (11)–(14) approximate the cost of traversing the cells corresponding to \(i, j \in V\) by considering the time-averaged cell intensities \(f_i, f_j\) over the approximate periods of time required to traverse these cells (namely, \(\Delta t_i\) and \(\Delta t_j\)). Stated differently, the edge cost function in Eqn. (11) is similar to the baseline cost function defined in Eqn. (2), considering that the sizes of the cells corresponding to \(i, j \in V\) are, respectively, \(|W(i, V)|\) and \(|W(j, V)|\), and that the time required to traverse these cells is approximately in proportion to the cell sizes.

A procedure to determine the locations and the sizes of cells in \(\Omega_{\text{mr}}\) in the vehicle-centric multiresolution approximation, including fast updates of these cell locations and sizes with the changing vehicle location, is provided in [13]. Furthermore, a procedure denoted MR-GRAPH to determine the edges in the graph \(\mathcal{G}\), including fast updates to the sets of vertices and edges of this graph with the changing vehicle location, is also provided in [13]. Due to the proposed use of time-varying wavelet coefficients, the topological properties (i.e., cell locations, sizes, and adjacency relations) of the vehicle-centric multiresolution cell decompositions in the proposed work are the same as in [13], and we may reuse MR-GRAPH described therein without modification.

The intensities of the cells in \(\Omega_{\text{mr}}\) are time-varying. Consequently, the cost of any edge in \(\mathcal{G}\) depends on the time at which the edge traversal occurs, as evident from Eqs. (11)–(14). Owing to the use of DWT in constructing the multiresolution cell decomposition, the computation of edge traversal costs in \(\mathcal{G}\) becomes easy. Specifically, by Eqn. (9), for any \((x, y) \in R\) and any \(t_1, t_2 \in T\),

\[
\int_{t_1}^{t_2} \hat{F}(x, y, s) \, ds = \sum_{k, \ell = 0}^{3} \sum_{n \in \mathbb{Z}_2} \tau_n \alpha^{n}_{m_0, k, \ell} \Phi^{m_0, k, \ell}(x, y) + \sum_{p=1}^{3} \sum_{m=m_0}^{2^{m_0}} \sum_{k, \ell = 0}^{2^{m_0}} \tau_n \beta^{p, n}_{m, k, \ell} \Psi^{p, n}_{m, k, \ell}(x, y),
\]

where \(\tau_n := \frac{(t_2 - t_0)^{n+1} - (t_1 - t_0)^{n+1}}{(n + 1)!}\). Equations (15) and (16) involve simple algebraic computations, and these equations may be used with minor modifications to compute the time-averages in Eqs. (12) and (13) required for evaluating edge costs in \(\mathcal{G}\).

Figure 2 shows in pseudo-code form the proposed path-planning algorithm based on the vehicle-centric multiresolution approximation of the spatial field \(\hat{F}\). The algorithm iterates Lines 3–10 until the goal is reached. At each iteration, the algorithm computes a vehicle-centric multiresolution approximation and the corresponding cell decomposition graph (Lines 1–4). In Line 5, the optimal path in \(\mathcal{G}_n\) is computed by an algorithm similar to that described in Fig. 1. The vehicle is assumed to traverse the first cell in the path \(\pi_n\), and the process is repeated for the new vehicle location.

For simplicity, the pseudo-code in Figure 2 omits two significant details. Firstly, the computations of \(\mathcal{A}_n\) and \(\mathcal{G}_n\) in Lines 3 and 4 need not be performed afresh at each iteration. Instead, \(\mathcal{A}_n\) and \(\mathcal{G}_n\) can be computed by making incremental changes to \(\mathcal{A}_{n-1}\) and \(\mathcal{G}_{n-1}\). The procedures for doing so are described in detail in [13]. We use these algorithms without modification in the proposed path-planning technique. Secondly, issues regarding repeated visits to vertices in \(\mathcal{G}\) at different iterations of the proposed path-planning algorithm (specifically, in Line 6) must be resolved. In [13], these issues are resolved by maintaining a record of the number of visits to each vertex in \(\mathcal{G}\) and by accordingly constraining the optimization problem in Line 5. We adopt a similar approach. Due to Assumption 5, the path optimization in Line 5 needs to maintain a fringe of vertices in \(\mathcal{G}_n\), instead of a fringe of time-vertex pairs. Consequently, the optimization in Line 5 simplified, and the algorithm in Fig. 2 shares convergence and completeness properties with the algorithm in [13]. We summarize the preceding developments with the main result of this paper as follows.

**Proposition 1.** The path-planning algorithm described in Fig. 2 solves Problem 2.

*Proof.* Omitted for brevity.

---

**Multiresolution Path-planning with Time-varying Costs**

- **Procedure MR-APPROX** (\(\bar{j}\))
  1. \(\mathcal{A} := \mathcal{A}^{\text{win}}(\text{cell}(j; \mathcal{G}))\) using Eqn. (10).

- **Procedure MAIN**
  1. \(\bar{x}_S, j_0 := \bar{x}_S, n := 0, \text{AtGoal} := 0, \bar{J}(\bar{\pi}) := 0\)
  2. **while** \(\neg \text{AtGoal}**
  3. \(\mathcal{A}_n := \text{MR-APPROX}(\bar{j}_n), \mathcal{G}_n := \text{MR-GRAPH}(\mathcal{A}_n)\)
  4. \(\pi_n := \arg \min \{\bar{J}(\pi) : \pi \text{ is a path in } \mathcal{G}_n\}\)
  5. \(\bar{j}_{n+1} := \text{vert}(\text{cell}(i_1; \mathcal{G}_n); \mathcal{G})\), where \(i_1\) is the second vertex in the path \(\pi_n\)
  6. \(\text{AtGoal} := (\bar{j}_{n+1} = \bar{x}_G)\)
  7. \(\bar{\pi} := (\bar{\pi}, \bar{j}_n)\)
  8. \(\bar{J}(\bar{\pi}) := \bar{J}(\bar{\pi}) + \bar{g}(\bar{j}_n, \bar{j}_{n+1}, n\delta t)\)
  9. \(n := n + 1\)

**Fig. 2.** Pseudo-code for the proposed path-planning algorithm.
IV. ILLUSTRATIVE EXAMPLE AND DISCUSSION

To illustrate the implementation of the proposed algorithm, we consider a time-varying spatial field that is described as the weighted sum of a finite number of time-invariant spatial fields, with time-varying weights. Specifically, we consider

\[
F(x, y, t) := \sum_{k=1}^{K} \left( \gamma_{k,0} + t \gamma_{k,1} F_k(x, y) \right),
\]

where \( \gamma_{k,0}, \gamma_{k,1} \in [0, 1] \) are constants, and \( F_k \) are time-invariant fields defined over \( \mathbb{R} \). Clearly, the field \( F \) satisfies Assumption 1. For this example, we choose \( K = 3 \). Figure 3 shows the intensity maps of the field \( F \) chosen for this example at different instants of time. In Fig. 3, the darkness is proportional to the field intensity. The size of the region \( \mathbb{R} \) is characterized by \( D = 7 \), i.e., each image consists of \( 2^D = 128 \) rows of pixels (the smallest square region over which the intensity is constant), with 128 pixels in each row.

We consider the path-planning problem with initial and goal vertices as indicated in Fig. 3(d), and we first solve Problem 1. The path found by solving Problem 1 by the algorithm in Fig. 1 is indicated in Fig. 3(d). In Figs. 3(a)–3(c), the red-colored dot indicates the location of the vehicle (arrows added for clarity) at the indicated time instants.

Next, we consider Problem 2 with the same initial and goal vertices. The images in Fig. 4 indicate the vehicle-centric multiresolution approximation of the field \( F \) constructed at various instants of time. In Fig. 4, the cells with red borders indicate the path found in Line 5 of the proposed algorithm. Notice that at each time instant, the field \( \hat{F} \) approximates \( F \) accurately in the immediate vicinity of the vehicle, and with lesser accuracy elsewhere.

Finally, Fig. 5 shows a comparison of the paths found in each of the preceding two cases (namely, by solution of Problem 1 with “complete information” and by solution of Problem 2 with vehicle-centric multiresolution information). Notice that the paths are not significantly different. In this particular example, the cost of the path obtained by using the vehicle-centric multiresolution approximation of the field was found to be 2% higher than the cost of the path found in the baseline “full information” case.

A thorough analysis of the computational benefits of the DWT-based multiresolution cell decomposition approach, compared to other multiresolution approaches, is available in [13]. Because the proposed approach shares graph topological properties with the approach in [13], the proposed approach also claims these computational benefits.

V. CONCLUSIONS

In this paper, we extended existing literature on wavelet transform-based motion-planning to allow the incorporation of traversal costs based on a time-varying spatial field. The proposed computations of time-varying DWT coefficients instead of three-dimensional (one temporal and two spatial) DWT coefficients are not only fast, but also allow the proposed approach to share important graph topological properties with the previously developed approach in [13]. We provided the details of calculating cell intensities, as well as the associated traversal costs, from the time-varying DWT coefficients. Future extensions of the proposed work include the allowance of “waiting” at any vertex of \( \hat{G} \), the allowance of cycles in the resultant path, and the previously discussed incorporation of vehicle dynamical constraints in...
For each \( m, k \in \mathbb{Z} \), we define scalar functions \( \phi_{m,k} \) and \( \psi_{m,k} \) by \( \phi_{m,k}(t) := \sqrt{2^m} \phi(2^m t - k) \), and \( \psi_{m,k}(t) := \sqrt{2^m} \psi(2^m t - k) \). The discrete wavelet transform of a scalar function \( f \in L^2(\mathbb{R}) \) is defined by \( a_{m,0,k} := \langle \phi_{m,k}(t), f(t) \rangle \), and \( d_{m,k} := \langle \psi_{m,k}(t), f(t) \rangle \), where \( m_0 \in \mathbb{Z} \). The 1D reconstruction equation is

\[
    f(t) = \sum_{k=-\infty}^{\infty} a_{m_0,k} \phi_{m_0,k}(t) + \sum_{m=m_0}^{\infty} \sum_{k=-\infty}^{\infty} d_{m,k} \psi_{m,k}(t).
\]

The scalars \( a_{m_0,k} \) and \( d_{m,k} \) are known as approximation and detail coefficients respectively. For the 2D extension of the 1D DWT, a scaling function \( \Phi_{m,k,l}(x,y) \) and three wavelets \( \psi_{m,k,l}^1, \psi_{m,k,l}^2, \psi_{m,k,l}^3 \) are defined. The 2D DWT coefficients of a scalar function \( f \in L^2(\mathbb{R}^2) \) are

\[
    a_{m_0,k,l} := \langle \Phi_{m_0,k,l}(x,y), F(x,y) \rangle, \quad (A.1)
\]

\[
    d_{m,k,l}^p := \langle \psi_{m,k,l}^p(x,y), F(x,y) \rangle, \quad (A.2)
\]

for \( p = 1, 2, 3, k, l \in \mathbb{Z} \), and \( m \geq m_0 \in \mathbb{Z} \). The 2D reconstruction equation is analogous to the 1D case.

An example of a pair of scaling function and wavelet is the Haar family [14]. For the 1-D Haar family, the functions \( \phi_{m,k} \) and \( \psi_{m,k} \) are compactly supported over the interval \( I_{m,k} := [2^{-m} k, 2^{-m} (k+1)] \), and by consequence, the functions \( \Phi_{m,k,l} \) and \( \Psi_{m,k,l} \) are compactly supported over

\[
    S_{m,k,l} := I_{m,k} \times I_{m,l}.
\]

REFERENCES


