Tetra

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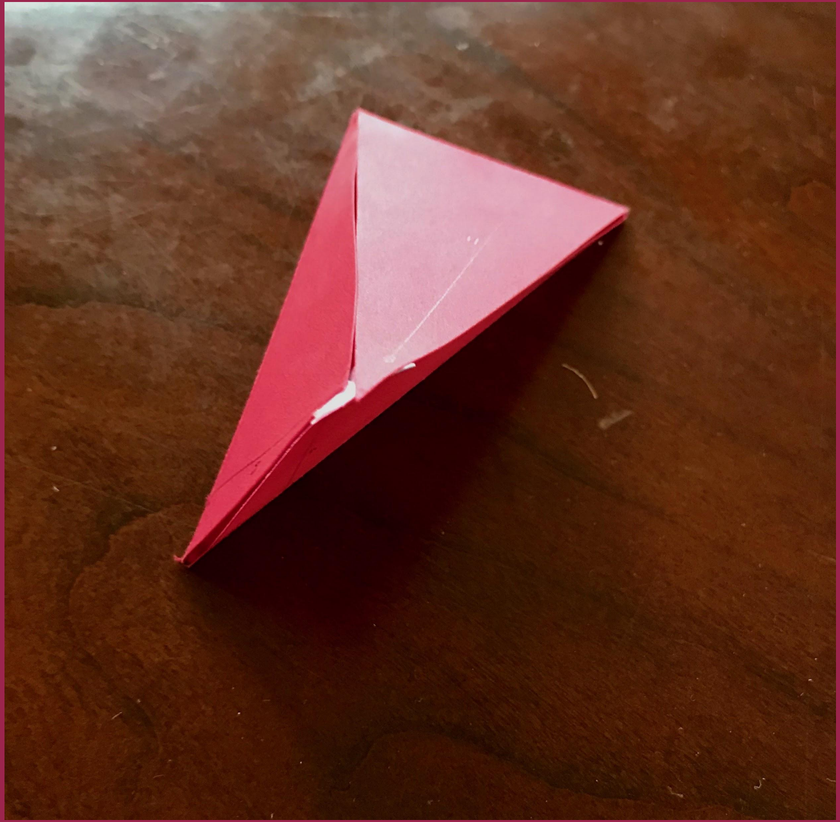
The Question:

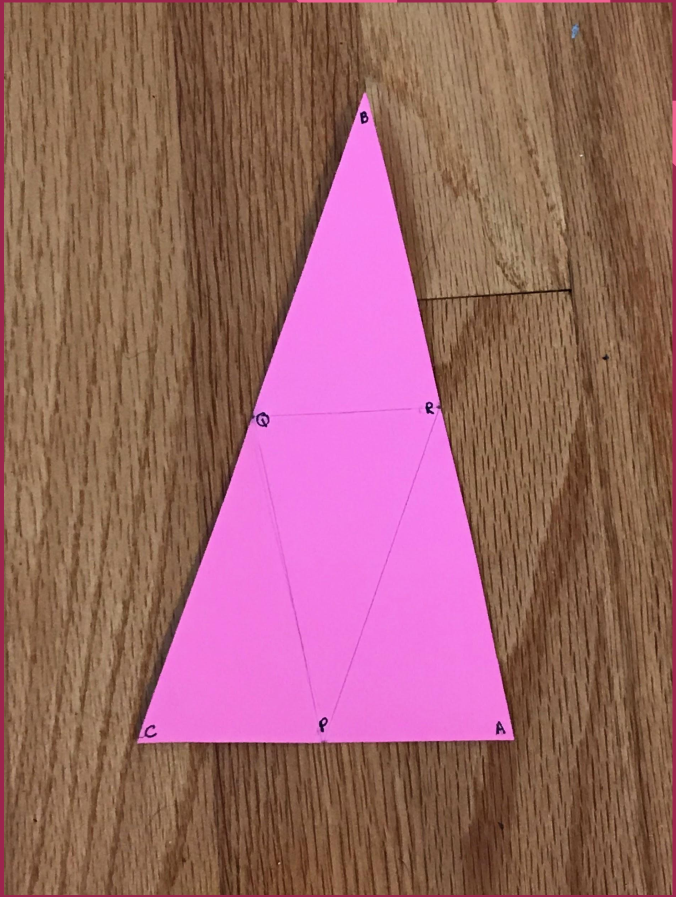
Tetrahedron Volume

The sides of triangle ABC measure 11 cm, 20 cm, and 21 cm. Let P, Q, and R be the midpoints of the sides. Fold along PQ, QR, and RP until vertices A, B, and C coincide. Calculate the volume of the resulting tetrahedron.

Method #1: 3D Angle

3D Model



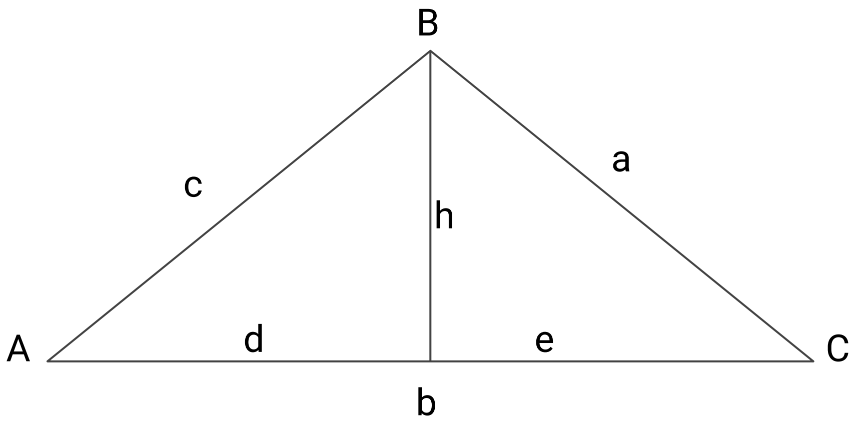


Law of Cosines

Deriving $a^2 = b^2 + c^2 - 2bc \cdot \text{Cos}A$

$a^2 = e^2 + h^2$
$c^2 = d^2 + h^2$
$h^2 = c^2 - d^2$
$b = d + e$
$d = e - b$
$d^2 = (e - b)^2$

$\text{Cos}A = d/c$
$d = c \cdot \text{Cos}A$



$a^2 = e^2 + h^2$
$a^2 = e^2 + (c^2 - d^2)$
$a^2 = (b - d)^2 + c^2 - d^2$
$a^2 = b^2 + d^2 - 2bd + c^2 - d^2$
$a^2 = b^2 + c^2 - 2bd$
$a^2 = b^2 + c^2 - 2bc \cdot \text{Cos}A$

Step 1: Angle Measures

Law of Cosines:

$$11^2 = 21^2 + 20^2 - 2(21)(20)\cos(\theta)$$

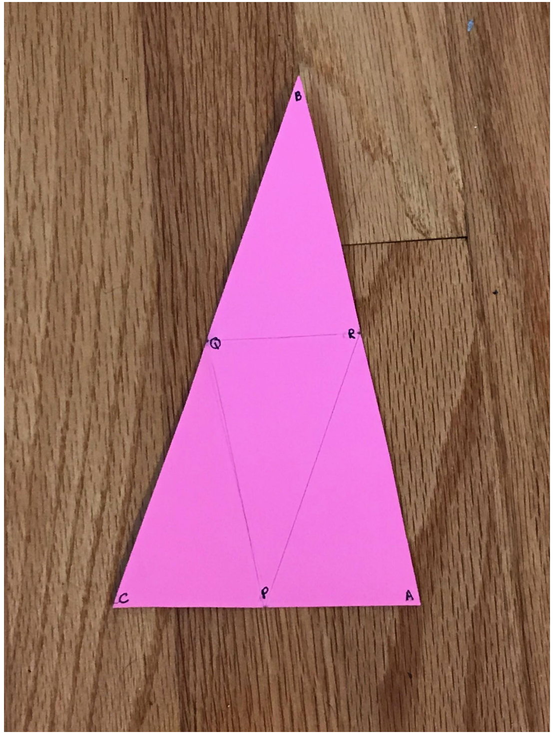
$$\theta = \cos^{-1}(6/7)$$

$$20^2 = 21^2 + 11^2 - 2(21)(11)\cos(\alpha)$$

$$\alpha = \cos^{-1}(27/77)$$

$$21^2 = 20^2 + 11^2 - 2(20)(11)\cos(\beta)$$

$$\beta = \cos^{-1}(2/11)$$



Step 2: Side lengths

They are all half of the original according to Triangle Midsegment Theorem.

AC = 11; AB = 20; BC = 21;

RQ = 11/2 = 5.5

PQ = 20/2 = 10

PR = 21/2 = 10.5

For Heron's Theorem:

a = RQ = 5.5cm; b = PQ = 10cm; c = PR = 10.5cm

Step 3: Heron's Theorem for Area of ΔPQR

[ΔPQR] = sqrt{s(s-a)(s-b)(s-c)}, where s is semiperimeter and a, b, and c are side lengths of ΔPQR

s = (5.5 + 10 + 10.5)/2

s = 13cm

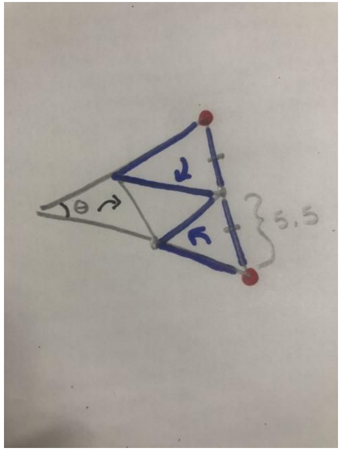
[ΔPQR] = sqrt{s(s-a)(s-b)(s-c)}

[ΔPQR] = sqrt{13(13-5.5)(13-10)(13-10.5)}

[ΔPQR] = sqrt{13(7.5)(3)(2.5)}

[ΔPQR] = sqrt(731.25) = 15*sqrt(13)/2

The Theta Lemma:



Imagine the tetra is being folded.

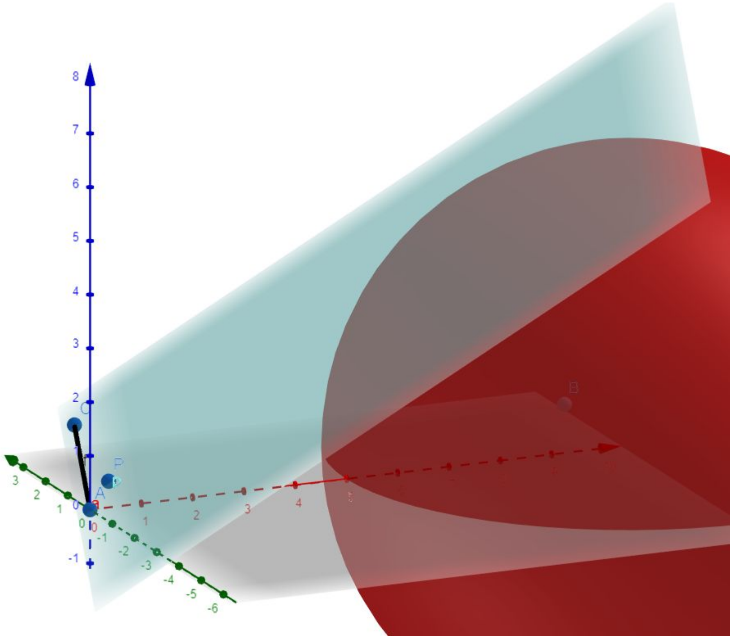
The red points will come together and the blue triangles will slowly become lines from a certain top view. We can imagine these blue lines as planes extending in and out of the paper. The angle between them (the 3D angle) is equivalent to the measure of the angle then of that corner of the face which comes to complete them.

Step 4: Jir Plane

In the following picture the Base of the Tetra (ABC) is shown on the XY plane. In respect to the AC line, Point P is shown to draw a plane at an angle measure of Theta from the XY plane into the positive Z dimension.

Why is this angle theta was explained on the previous slide.

The red ball is the model of all points where the endpoint of the 5.5 edge of the tetrahedron may rest.



Step 5: Jir Equations

Point P:

$$P = \left(\cos\left(90 - \cos^{-1}\left(\frac{27}{77}\right)\right), \sin\left(90 - \cos^{-1}\left(\frac{27}{77}\right)\right)\boxed{W}\sin\left(\cos^{-1}\left(\frac{6}{7}\right)\right)\right)$$

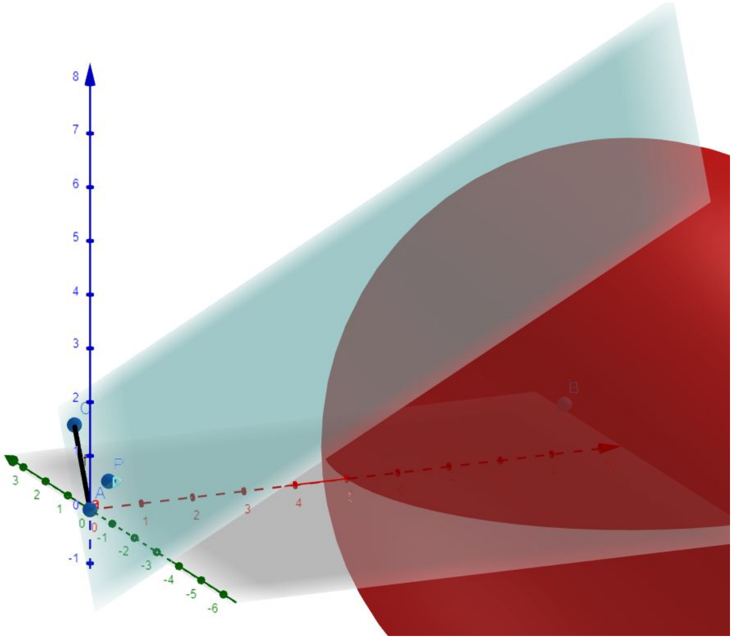
Plane: $p : \text{Plane}(A, C, P)$

$$\rightarrow \; 1.32x - 0.5y - 2.09z = 0$$

Red Ball:

$$a : y^2 + (x - 10.5)^2 + z^2 = 5.5^2$$

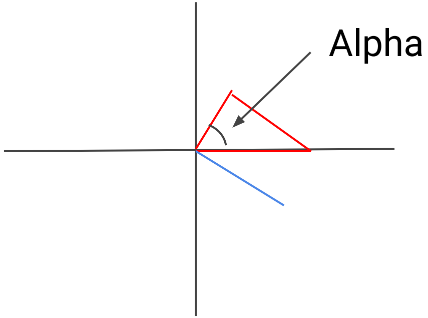
$$\rightarrow \; (x - 10.5)^2 + y^2 + z^2 = 30.25$$



The equation of a sphere is: $(x-h)^2+(y-k)^2+(z-j)^2=r^2$

W = 18*sqrt(13)/130

Why is Point P Point P?

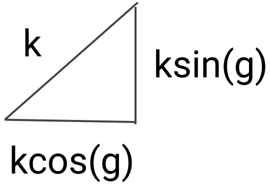


X = cosine to get X position

Y = sine to get Y position

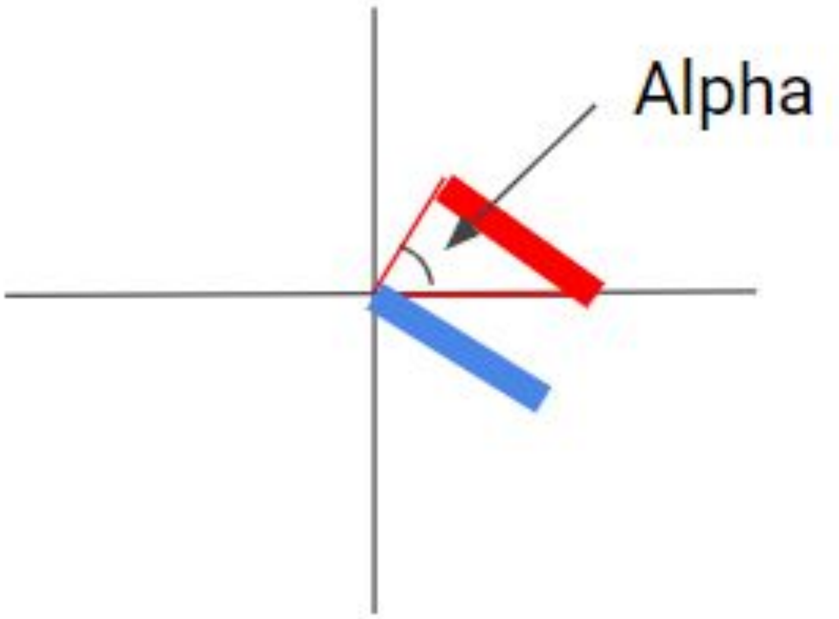
Z = Includes theta because this angle only exists in the 2D plane

$$P = \left(\cos\left(90 - \cos^{-1}\left(\frac{27}{77}\right)\right), \sin\left(90 - \cos^{-1}\left(\frac{27}{77}\right)\right), 0.5\ \sin\left(\cos^{-1}\left(\frac{6}{7}\right)\right)\right)$$



Slides 6: W >> H

18*sqrt(13)/130 AKA W is in relation to the 10 side. We have to multiply by 10 for it to be the height.



H = 18*sqrt(13)/13

A = 15*sqrt(13)/2

H*A/3 = 45 cm^3

Method #2: Intersection of Spheres

Step 1: Graph Points

Let point P be placed on the origin: (0,0,0)

Then point R could be placed on (10.5,0,0)

Point Q then will be (x_1,y_1,0)

Point V (the vertex) will be (x_2,y_2,z_2). (Will be the point of intersection of points A, B, and C.)

Step 2: Angle Measures

Law of Cosines:

$$5.5^2 = 10.5^2 + 10^2 - 2(10.5)(10)\cos(\theta)$$

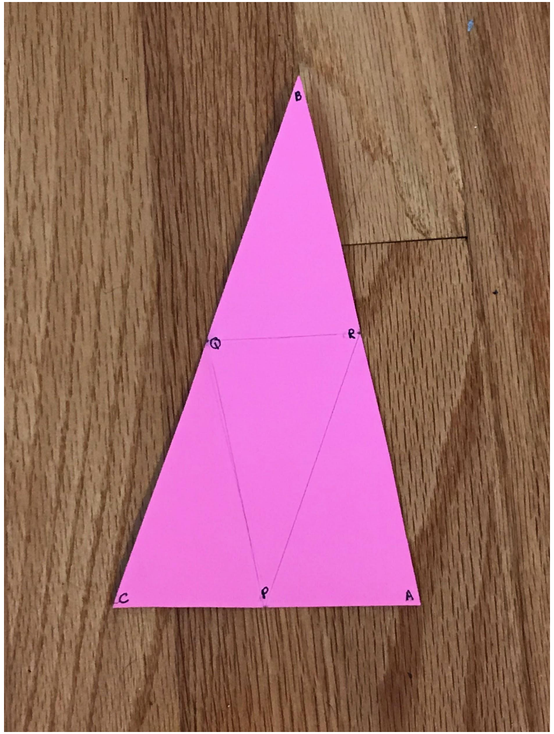
$$\theta = \cos^{-1}(6/7) = 31.002 \text{ degrees}$$

$$10^2 = 10.5^2 + 5.5^2 - 2(10.5)(5.5)\cos(\alpha)$$

$$\alpha = \cos^{-1}(27/77) = 69.472 \text{ degrees}$$

$$10.5^2 = 10^2 + 5.5^2 - 2(10)(5.5)\cos(\beta)$$

$$\beta = \cos^{-1}(2/11) = 79.524 \text{ degrees}$$



Step 3: Coordinates of Q

cos (31.002) = x1/10

x1 = 8.571

sin(31.002) = y1/10

y1= 5.151

Therefore, coordinates of Q are: (8.571,5.151,0)

Step 4: Construction of Spheres

The edges of the tetrahedron could act as the radii of the spheres.

We know PV = 5.5; QV = 10.5, RV = 10 (Since AP = 5.5, BQ =10.5, AR = 10)

The equation of a sphere is: $(x-h)^2+(y-k)^2+(z-j)^2=r^2$

By this and the coordinates of points P, Q, and R (as the center point of the sphere) we know:

$(x-0)^2+(y-0)^2+(z-0)^2=5.5^2$

$(x-8.571)^2+(y-5.151)^2+z^2=10.5^2$

$(x-10.5)^2+(y-0)^2+(z-0)^2=10^2$

Step 5: Solving System of Equations

Eq1: $(x-0)^2+(y-0)^2+(z-0)^2=5.5^2 \rightarrow x^2+y^2+z^2=5.5^2$

Eq2: $(x-8.571)^2+(y-5.151)^2+z^2=10.5^2$

Eq3: $(x-10.5)^2+(y-0)^2+(z-0)^2=10^2 \rightarrow (x-10.5)^2+y^2+z^2=10^2$

$x = 1.928$

$y = -1.269$

$z = 4.992$

$\therefore$Coordinates of V = (1.928, -1.269, 4.992)

Work Needed for Solving Equations

Eq1: $x^2+y^2+z^2=5.5^2$

Eq2: $(x-8.57)^2+(y-5.15)^2+z^2=10.5^2$

Eq3: $(x-10.5)^2+y^2+z^2=10^2$

$$\left.\begin{aligned} &x^2+y^2+z^2=5.5^2 \\ -\,&(x-10.5)^2+y^2+z^2=10^2 \end{aligned}\right\} \text{Eq 1 - Eq 3}$$

$$x^2-(x-10.5)^2=5.5^2-10^2$$
$$x^2-(x^2-21x+110.25)=-69.75$$
$$\cancel{x^2}-\cancel{x^2}+21x-110.25=-69.75$$
$$21x-110.25=-69.75$$
$$21x=40.5$$
$$x=\cancel{1.928}\ 1.928$$

$(1.928)^2+y^2+z^2=5.5^2$ Sub x-value in Eq 1

$(1.928-8.57)^2+(y-5.15)^2+z^2=10.5^2$ Sub x-value in Eq 2

$$3.717184+y^2+z^2=30.25$$
$$y^2+z^2=26.532816$$

$$44.116164+(y-5.15)^2+z^2=110.25$$
$$(y-5.15)^2+z^2=66.133836$$

$$\begin{aligned} &y^2+z^2=26.532816 \\ -\,&(y-5.15)^2+z^2=66.133836 \end{aligned}$$

$$y^2-(y-5.15)^2=-39.60102$$
$$y^2-(y^2-10.3y+26.5225)=-39.60102$$
$$\cancel{y^2}-\cancel{y^2}+10.3y-26.5225=-39.60102$$
$$10.3y=-13.07852$$
$$y=-1.269$$

$(1.928)^2+(-1.269)^2+z^2=5.5^2$ Sub x-value, y-value in Eq1

$$z^2=(5.5)^2-(1.928)^2-(-1.269)^2$$
$$z^2=24.922$$
$$z=\pm 4.992$$

$$(x,y,z)=(1.928,-1.269,\pm 4.992)$$

Step 6: Finding Area of Base

[ΔPQR] = sqrt{s(s-a)(s-b)(s-c)}, where s is semiperimeter and a, b, and c are side lengths of ΔPQR

s = (5.5 + 10 + 10.5)/2

s = 13cm

[ΔPQR] = sqrt{s(s-a)(s-b)(s-c)}

[ΔPQR] = sqrt{13(13-5.5)(13-10)(13-10.5)}

[ΔPQR] = sqrt{13(7.5)(3)(2.5)}

[ΔPQR] = sqrt(13*(225/4)) = 15*sqrt(13)/2

Step 7: Calculating Volume of Tetrahedron

Volume of tetrahedron = ⅓*b*h

Area of base = 15*sqrt(13)/2

Height = 4.992 (since z coordinate of Vertex would equal to height of the tetrahedron)

Volume = ⅓*15sqrt(13)/2*4.992 = 44.997 ≈ 45 cm^3

Any Questions?