Instructor's Notes to Accompany

Differential Equations: Modeling with MATLAB

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Contents

1 Overview and structure **4**

2 Learning aids in the text **5**
- 2.1 Exercises . . . 5
- 2.2 Exercise guides . . . 5
- 2.3 *Stop and think* . . . 6
- 2.4 Solutions to exercises . . . 6
- 2.5 Chapter projects . . . 6
- 2.6 Quick Reference Guide . . . 7
- 2.7 Matlab notes . . . 7

3 Syllabus suggestions **8**
- 3.1 Modeling . . . 8
  - 3.1.1 Core models . . . 10
  - 3.1.2 Other categories of models . . . 15
  - 3.1.3 Maximal modeling . . . 15
  - 3.1.4 Linking modeling to ideas . . . 17
  - 3.1.5 My preferences . . . 17
  - 3.1.6 Minimal modeling . . . 22
- 3.2 Numerical methods . . . 23
  - 3.2.1 Overview . . . 23
  - 3.2.2 Maximal numerical methods . . . 24
  - 3.2.3 Minimal numerical methods . . . 26
- 3.3 Analytical methods . . . 26
  - 3.3.1 Overview . . . 27
  - 3.3.2 Maximal analytic solutions . . . 29
  - 3.3.3 Minimal analytic solutions . . . 31
- 3.4 Graphs and phase diagrams . . . 32
- 3.5 Partial differential equations . . . 32

4 Section by section emphasis **33**

5 Software **33**

6 Instructor aids **34**

List of Tables

1 Core Models 11
2 Other Models 16
3 Ideas Used in Analyzing First-Order Models 18
4 Ideas Used in Analyzing Higher-Order Models 19
5 Ideas Used in Analyzing Higher-Order Models (cont'd) 20
6 Primary Emphasis of Each Section 34

1 Overview and structure

This text uses several complementary, spiral approaches.

- Models of physical problems motivate differential equations. The differential equations demand analytical, numerical, and graphical tools for their analysis. Interpreting the results of the analysis leads back to the physical problem, which then demands deeper analysis or better models that themselves require more sophisticated analysis.

  These distinct steps are often identified explicitly by section titles or by marginal labels like *Model*, *Analysis*, or *Interpretation*. For example, subsections 2.1.3, 2.1.4, and 2.1.5, p. 30–34, successively derive the simple population model, analyze it, and interpret the analysis; the subsection titles are A Model, Analysis, and Interpretation, respectively. Marginal notes on p. 2–3 identify the modeling, analysis, and interpretation steps for the projectile model considered there.

- Most mathematical ideas are introduced intuitively before they are defined formally, so that definitions can arise naturally, rather than appear as arbitrary rules.

- To give students the experience of generalization, common methods and ideas are introduced in succession for first-order scalar equations, then for second-order equations, then for first-order systems. For example, characteristic equations appear in all three settings, and each reappearance of the problem of solving a constant-coefficient, homogeneous, linear equation turns back to a previous, simpler problem for guidance in attacking the newer, more complex problem.

  > Of course, the mathematician in me is impatient to show the power of the general approach. However, I have found that such haste keeps all but the best students from seeing the power of generalization. Climbing the mountain step-by-step gives a better appreciation for the view than traveling to the summit for the first time via helicopter!

Analytical, graphical, and numerical tools are all introduced early in anticipation of later refinement, extension, and generalization. Chapter 1, Prologue, surveys this mix, giving a sample of modeling, of finding an analytic solution of an initial-value problem via separation of variables, of a numerical approximation using Euler's method, and of direction fields and solution graphs. Subsequent chapters are more specialized, as summarized in table 6.

> This approach could be viewed as the "Rule of Three Plus One", the analytical, graphical, and numerical perspectives augmented by the physical. See P. W. Davis, Asking Good Questions about Differential Equations, *College Math. J.*, **25**(5) 1994, 394-400.

2 Learning aids in the text

Learning aids built into the text include the following:

2.1 Exercises

Exercises appear at the end of every section (with the exception of the sections in chapter 1, Prologue), and a comprehensive set of exercises concludes every chapter. In total, there are more than 1,350 exercises, ranging from trivial mechanical exercises to difficult derivations, proofs, and analyses.

Many of the more challenging exercises provide too little data or too much data. Or they ask for explanations or plausibility arguments or similarities or differences. All are substantial departures from the "plug 'n' chug" exercises with unique answers to which many students are accustomed. In recompense, some students will also discover that mathematics is much more interesting than they thought.

2.2 Exercise guides

Exercise guides at the beginning of each set of exercises suggest particular problems to which students can turn in order to practice certain skills or to improve their understanding of particular ideas. To use these simple tables effectively, students must understand the vocabulary of the subject, and they must connect the vocabulary to what they know.

Consequently, the ostensible purpose of these guides—mapping course concepts into exercises—is realized only for those students who understand the elements in the domain of this mapping. To use a different metaphor, students must distinguish among the trees in the forest before an exercise guide becomes a useful map through it. Hence, the subversive purpose of the exercise guides is fostering understanding through mastery of vocabulary. Students who know enough to solve the inverse problem—using an exercise guide to determine which ideas apply to a given problem—already understand enough to make the exercise guide irrelevant.

> I sometimes suggest that students review for a test as follows: For each of the techniques or concepts listed in the left-hand column of an exercise guide, answer such questions as, "Is this a method or a concept? If it is a method, when it is applicable and how does it work? If it is a concept, how is it defined? When and why is it useful?"

2.3 *Stop and think*

Stop and think questions and challenges are sprinkled throughout the text to provoke thought while students are reading, to support teaching assistants or others who are conducting discussion sections, and to provide supplementary homework or in-class writing assignments. For easy reference, each of the more than 380 *Stop and think*'s is individually numbered within each chapter. They are designed to help students learn to read mathematics the way mathematicians do, with pencil in hand and with "brain open", as Paul Erdös might have said.

2.4 Solutions to exercises

Solutions for most odd-numbered exercises are provided on pages 644–678. Solutions are also given for an occasional even-numbered exercise whose solution parallels that of an odd-numbered exercise. Many of these solutions are given in some detail, as if they were terse examples augmenting the more than 250 conventional examples that appear in the text.

2.5 Chapter projects

Chapter projects at the end of each chapter are assignments that can be tackled by teams of two to four students over a period of ten days or so. These forty open-ended challenges often have several reasonable solutions and no single really satisfactory solution. (Proofs, of course, are an exception.) Projects like these can provoke some deep involvement with mathematics, an experience that many students have never had, as well as requiring careful oral and/or written communication of the students' final problem statement, their analysis, and their results, depending on what you decide to assign.

> For additional projects, some suggestions on ways to use projects in class, and other support materials, visit http://www.wpi.edu/ heinrich/odeproj/odeproj.html

2.6 Quick Reference Guide

The four pages on the inside front and back covers and the facing flyleaf pages contain a Quick Reference Guide to the methods and concepts covered in the text. It is another tool for getting a view of the forest through all of the trees. Although this quick summary suffers from omission and simplification, as does any such condensation, students can challenge themselves to explain each of these ideas listed there, to construct examples using each of the methods, and to explain connections among the various entries.

> For a quick answer to the canonical question "What will be on the test?", put check marks next to the appropriate items on a copy of the four pages of the Quick Reference Guide and post it on your office door!

2.7 Matlab notes

Boxes at various points in the text labeled "MATLAB" pose a question or suggest an activity that uses MATLAB or the graphical user interface DELAB (see section 5 below) to illuminate the ideas under consideration.

These are positioned to be unobtrusive but in the right place at the right time. Ideally, a student is reading next to a computer, visualizing, exploring, discovering, making notes and thinking deeply about each new idea as it is encountered. Since the reality is often a bit different, these computational and visual references are structured so that not completing them won't stop the student. Ultimately, students will give these MATLAB activities the same importance as you do through using them as homework, as laboratory exercises, as demonstrations in class, and so on.

Many of the homework exercises specifically ask for comparison with or exercise of a specific tool built into DELAB; e.g., part of an exercise might ask whether the analytic solution tool in DELAB produces the same solution formula as undetermined coefficients, say, or whether it produces a general solution. Or students might be asked to use DELab to verify their graphical or analytic study of a given equation and to comment on the differences. These kinds of questions are meant to nurture a skeptical understanding of these computational tools while nurturing good judgment and flexibility in their use.

3 Syllabus suggestions

Rather than attempt to offer semester-long sample curricula, the following sections suggest which chapters and sections work best to accomplish particular ends. From these alternatives, individual instructors can then organize the particular journey they want their students to take.

3.1 Modeling

This text begins with models and uses them throughout. But few of us have the time to pursue every model in full detail through the complete cycle of derivation, analysis, and interpretation. The following are suggestions for three different approaches, listed in increasing order of course time required.

> My own approach is to mix the three approaches, using less and less class time for modeling and depending more and more on the student own work as the term progresses. The expectation that students will assume increasing responsibility for their own learning as the course progresses is reinforced by class discussions in lecture and recitation sections, by the homework exercises assigned, and by the questions posed on sample and actual examinations.

1. **"Take my word for it":** Simply state the governing equations, and perhaps assign reading of the relevant pages from the text, without further class discussion. This approach is not satisfactory as the *only* introduction to mathematical modeling, but it is appropriate once students have gained some experience with modeling.

2. **Plausibility argument:** State the governing equations. Then give a plausibility argument for the terms in the equation, or lead students to develop their own arguments through a class discussion, or assign exercises that require such an argument.

For example, the text only provides a plausibility argument for the competition equations (2.33–2.34)

$$\begin{aligned} y' &= y(1 - y - ez) \\ z' &= rz(1 - z - fy) \end{aligned}$$

in subsection 2.4.3, p. 72. Then exercise 5 of section 2.4, p. 75, guides students through a step-by-step derivation.

To provoke class discussion that builds intuitive confidence in a model, consider using *Stop and think* questions such as 2.21, p. 67, or 2.24 and 2.25, p. 70.

3. **First principles** Mimicking the derivation of the model of vertical motion in section 1.2, p. 1–5, or the simple population model in section 2.1, p. 27–31, introduce and motivate the relevant experimental facts and observations (e.g., force of gravity is proportional to mass), introduce the governing physical law (e.g., $F = ma$), then derive the differential equation(s) and initial condition(s).

   > Many of the models in the text are derived in this relatively careful manner, but I think it a mistake to plod through every single equation in such detail in class. Instead, some can be derived and discussed carefully in class, some can be stated with a quick plausibility argument, and others can rely on students "taking your word for it." The text is available for back-up reading assignments, and the importance given such reading assignments will be determined by the homework exercises you assign and the test questions you ask.

   As examples of exercises that support the middle road, a plausibility argument for a model, consider:

   - *Chapter 2 chapter exercises, exercise 15, p. 79:* An insect population (denoted by y) is being destroyed by an insecticide at a constant rate d. It is increasing at a rate proportional to the current population; the constant of proportionality is g. Which of the following initial-value problems might be a reasonable model of this situation? Justify your rejection of each unacceptable choice.
     (a) $y' + gy = d,\ y(0) = y_i$

(b) $y' - gy = d,\ y(0) = y_i$

(c) $y' + gy = -d,\ y(0) = y_i$

(d) $y' - gy = -d,\ y(0) = y_i$

- *Section 5.1, exercise 7, p.217:* The text derived the model

$$mx'' + kx = k(h(t) - h(0)),\ x(0) = x_i,\ x'(0) = v_i,$$

for the forced, undamped vertical spring-mass system. It also considered a spring-mass system without forcing that was damped by friction. It obtained the model

$$mx'' + px' + kx = 0,\ x(0) = x_i,\ x'(0) = v_i\,.$$

In fact, even a freely suspended mass is subject to damping due to air resistance and internal resistance in the spring. These damping forces can be modeled as being proportional to velocity and acting in the opposite direction, just like the force of friction acting on the mass sliding horizontally.

(a) Compare the undamped, forced model with the damped model without forcing. Predict what a model of a forced, damped system would look like.

(b) Verify your prediction by deriving such a model. Specifically, consider the system shown in figure 5.5 and suppose that the mass is subject to air resistance which is proportional to its velocity and acts in the opposite direction.

3.1.1 Core models

Several models are used repeatedly in one or more chapters to provide a common reference point for the introduction and testing of new methods and ideas. To help you decide how to allocate your time, these core models, their point of first introduction, and their subsequent appearances are described below (see p. 12) and summarized in table 1.

As suggested in the preceding section, your options for depth of coverage range from "Take my word for it." to deriving each model in painstaking detail. I advocate the middle ground, plausibility arguments with back-up reading and homework assignments, in addition to a few complete derivations and a few models issued by decree from higher authority.

Core Models

Model	Introduced in section	Used in sections
Projectile motion		
Rock model: $v' = -g$	1.2	1.3; 3.2; 4.1, 4.4; 6.6
Scalar population		
Simple: $P' = kP$	2.1	3.1–3.2; 4.1–4.3
Emigration: $P' = kP - E$	2.2.1	3.2–3.3; 4.2–4.4 10.4–10.5
Logistic: $P' = aP - sP^2$	2.2.2	3.2–3.3; 4.1–4.3
Heat flow	2.3	3.1–3.3; 4.1, various
$T' = -(Ak/cm)(T - T_{\text{out}})$		exercises in 4.2–4.5 10.4; 10.5, exercise 5
Multi-species population		
Predator-prey (2.24–2.25), p. 67	2.4.1	7.1, exercises 2–3 7.2, exercises 12–13 8.1; 8.3, exercises 9, 33
Epidemic (SIR, SIRS) (2.29–2.30), p. 71	2.4.2	7.1, exercises 2–3 7.2.3; 7.2, exercises 9–11 8.1; 8.3, exercises 9, 26 cover of text
Competition (2.35–2.36), p. 73	2.4.3	7.1, exercises 2–3 7.2, exercises 14–17 8.3, exercise 9
Mechanical oscillators		
Spring-mass: $mx'' + px' + kx = F(t)$	5.1	6.1, 6.4–6.7 10.5, exercises 6, 10
Pendulum: $L\theta'' + g\sin\theta = 0$	5.2	6.1, 6.5.2, 6.8 7.1–7.2; 8.3 10.5, exercise 9
Steady diffusion		
$-D(Ac')' + (AVc)' = 0$	9.1	9.2–9.3; 11.2
Time-dependent diffusion	9.4	9.5–9.6
$\partial T/\partial t = \kappa \partial T^2/\partial x^2$		11.5; 11.7, exercise 25

Table 1: Core models used in the text, their point of first introduction, and sections where they are subsequently used. See the discussion beginning on page 12 of this manual for more detail.

Briefly, the first-order scalar population models (Malthus, emigration, logistic) and heat-flow derived in chapter 2, Models from Conservation Laws, appear repeatedly in chapters 3 and 4, which develop numerical, graphical, and analytical ideas for first-order scalar equations. The simple harmonic oscillator (spring-mass model) and the linear and nonlinear pendulum derived in sections 5.1 and 5.2 are the standard examples in chapter 6, which treats second-order analytic solution methods and applies numerical methods. The pendulum and the SIRS (epidemic) system (subsection 2.4.2) motivate phase plane analysis in chapter 7, Graphical Tools for Two Dimensions, and linearized phase plane analysis in section 8.3, Connections with the Phase Plane (after sections 8.1 and 8.2 have solved constant-coefficient, homogeneous linear systems.)

Second-order boundary-value problems, ordinary and partial respectively, build upon the steady-state diffusion model of section 9.1, Diffusion Models, and upon the heat equation developed in section 9.4, Time-dependent Diffusion.

Chapter 10, The Laplace Transform, uses as examples linear first- and second-order initial-value problems involving population and heat-flow from chapter 2 and spring-mass systems from section 5.1. Some of the additional analytic methods of chapter 11, More Analytic Tools for Two Dimensions, also draw upon the diffusion equations of sections 9.1 and 9.4 for examples of equations with nonconstant coefficients or singular points.

For more detail, the point of first introduction, and subsequent uses of the primary models, see table 1 or the following paragraphs. For an overview of the connections between mathematical ideas and models, see tables 3, 4, and 5.

First-order scalar population models The simple (Malthusian) model ($P' = kP$, section 2.1, p. 27–36), the emigration model ($P' = kP - E$, subsection 2.2.1, p. 41–46), and the logistic equation ($P' = aP - sP^2$, subsection 2.2.2, p. 57–50) are used as the standard examples of first-order equations that are linear and homogeneous, linear and nonhomogeneous, and nonlinear, respectively. They are used throughout chapters 3 and 4 to introduce analytical ideas and methods, numerical methods, and graphical analyses.

The logistic equation is the standard nonlinear foil to linear equations. In particular, it is the test bed for more complex graphical analysis (section 3.2, Direction Fields and Phase Lines, p. 108–118), multiple steady states and linearized stability analysis (section 3.3, Steady States, Stability, and Linearization, p. 118–126), the limitations of analytic methods (e.g., separation of variables does not find the solution $P = a/s$, example 19, p. 152), and the significance of uniqueness (example 46, p. 190).

The emigration model, $P' = kP - E(t)$ with E defined appropriately, motivates the study of the Laplace transform of both the unit step function and the Dirac delta function in sections 10.4, Ramps and Jumps, p. 546–553, and 10.5, The Unit Impulse Function, p. 556–559.

Heat-flow model The heat-flow model $T' = -(Ak/cm)(T - T_{\text{out}})$ of section 2.3, p. 56–61, is the first-order, linear, nonhomogeneous alternative to the emigration model $P' = kP - E$. It appears throughout chapters 3 and 4 to illustrate numerical methods, steady states, and the various analytic solution methods.

Although its derivation involves possibly elusive notions like energy, thermal conductivity, and specific heat, it is a natural candidate for a plausibility argument:

> The temperature of the body falls if it is hotter than its surroundings ($T > T_{\text{out}}$). The rate of decrease of temperature increases if there is more surface area A to transfer heat energy or if thermal conductivity k is larger (i.e., if insulation is poorer).

The concept of heat flow appears again in section 9.4, Time-dependent Diffusion, p. 460–467, where the heat equation $T_t = \kappa T_{xx}$ is derived.

Multiple species models In varying degrees of detail, section 2.4, p. 65–73, derives three standard multiple-species models. They are predator-prey (fox-rabbit),

$$\begin{aligned} F' &= -(d_F - \alpha R)F \\ R' &= (b_R - \beta F)R, \end{aligned}$$

epidemic (SIRS),

$$\begin{aligned} S' &= -bIS + g(P - S - I) \\ I' &= bIS - rI, \end{aligned}$$

and competition,

$$\begin{aligned} y' &= y(1 - y - ez) \\ z' &= rz(1 - z - fy). \end{aligned}$$

As these models are introduced, the text briefly considers relevant phase plane and equilibrium ideas.

A detailed study of the phase plane begins in chapter 7, Graphical Tools for Two Dimensions, using the pendulum model developed in section 5.2. The SIRS model joins the pendulum in section 7.2, Nullclines and Local Linearization, p. 343–350, and one of its phase plane diagrams appears on the cover of the book.

All three multiple-species models appear in the exercises for section 7.2, Nullclines and Local Linearization, for section 8.1, Basic Definitions: Systems, and for section 8.3, Connections with the Phase Plane. As appropriate, these exercises request nullcline analyses and phase plane sketches, or they request the determination of equilibria and a linearized phase plane analysis about each. Students can reasonably be expected to complete such exercises, including providing a physical interpretation of the results, based on a prior exposure to a plausibility argument for just one of these models, say SIRS.

> I usually do not derive any of these systems in class. Rather, I introduce them as needed to illustrate a point, providing a plausibility argument for the terms in each model when they are introduced.

Oscillatory models The classic spring-mass harmonic oscillator equation $mx'' + px' + kx = F(t)$ and its equivalent first-order system are derived in section 5.1, p. 203–215, the linear and nonlinear pendulum equations ($L\theta'' + g\theta = 0$, etc.) and their equivalent first-order systems in section 5.2, p. 219–221. The linear spring-mass model is used extensively in chapter 6 Analytic Tools for Two Dimensions, which develops the usual second-order analytic solution methods and applies them to analyze the behavior of these systems.

The nonlinear pendulum equation $L\theta'' + p\theta' + g\sin\theta = 0$ is the nonlinear foil to the simple harmonic oscillator for the study of equilibria, linearization, nullclines, stability, etc. It and its linear relative are used repeatedly in section 6.8, Linear versus Nonlinear, p. 321–325, in section 7.1, The Phase Plane, p. 334–339, in section 7.2, Nullclines and Local Linearization, p.

343–350, and in section 8.3, Connections with the Phase Plane, the study of linearized phase plane analysis.

Diffusion models, ordinary and partial The stationary diffusion equation with convection but no source, $-D(A(x)c'(x))'+(A(x)V(x)c(x))'=0$, is derived in section 9.1, Diffusion Models, p. 428–433. Derivations of various thermal analogs are requested in section 9.1, exercises 9–14. Diffusion models provide the examples used in sections 9.2, Boundary-value Problems: Analytic Tools, and 9.3, Boundary-value Problems: Numerical Methods. The equation for diffusion in a circular domain (e.g., example 2, p. 432) is used in section 11.2, Cauchy-Euler Equations, as an example of such an equation.

The heat equation $\partial T/\partial t = \kappa \partial T^2/\partial x^2$ is derived in section 9.4, Time-Dependent Diffusion, to provide the motivating example for the two subsequent sections, 9.5, Fourier Methods, and section 9.6, Initial-Boundary-Value Problems: Numerical Methods. Separating the heat equation on a circular domain also provides an example of a Bessel equation for section 11.5, Regular Singular Points.

3.1.2 Other categories of models

A number of other models are introduced as well, but their use is localized or confined primarily to exercises or to projects.

The one exception to local use is *RLC* circuit equations, which are analyzed in separate sections as indicated in table 2. These analyses parallel those for the two mechanical oscillators—spring-mass systems and the linearized pendulum. *RLC* models included for those who want to emphasize electrical models, but they can be omitted entirely without difficulty.

3.1.3 Maximal modeling

Even if you commit to covering *every* model in the text (and that's a lot!), you won't want to derive every single one in lecture in complete detail. Instead, shift more and more of the responsibility to the students as the course progress by using the approaches (plausibility arguments, "Take my word for it" followed by reading assignments, etc.) suggested in section 3.1 of this manual.

With such a balance in mind, a maximal-modeling approach to constructing a syllabus would cover *every* section listed in the center column of tables

Other Models

Model	Introduced in section	Used in sections
Circuits		
RC circuits	2.5, exercises 3–6	
RLC circuits	5.3	6.5.3, 6.7.2; 7.3
van der Pol	6.10, project 3; 7.3	7.3
Various first-order		
Radioactive decay	2.5, exercises 1–2, 11	
Mixing	3.4, exercise 3	
Continuous compound interest	2.5, exercise 14	
Other single-species	2.5, exercises 15, 19	
populations	3.5, project 6	
Chemical, biological reactions	2.5, exercises 8, 13	
Logistic map	3.5, project 3	
Time-delayed logistic	10.7, project 2	
Nonlinear oscillators		
Nonlinear springs	5.5, projects 1–2	
van der Pol	6.10, project 3; 7.3	7.3
Various systems		
Autocatalytic reactions	7.5, project 1	
Other predator-prey	7.5, project 2	
Chemostat	8.6, project 4	

Table 2: Other models introduced in the text.

1 and 2 through a lecture, a class discussion, homework problems, or a reading assignment. Such an ambitious undertaking would spill well beyond a single semester!

3.1.4 Linking modeling to ideas

One corner of the modeling triangle is deriving mathematical statements from experimental observations and physical laws. Another is using mathematical concepts—analytical, graphical, and numerical—to analyze the behavior predicted by the model. The third is interpreting that behavior in light of the original physical problem. Tables 3, 4, and 5 summarize the mathematical concepts that are associated with the analysis and interpretation of many of the models introduced in this text.

From a modeling point of view, the best approach to constructing a syllabus is to decide which ideas you regard as most important and which models are likely to be of most interest to you and your students. Use tables 3, 4, and 5 to strike the balance that is best for you—the left-hand columns of those tables list models, the right-hand columns list the ideas that are introduced using those differential equations.

The time you spend in your course with analysis and interpretation is time in part devoted to modeling and time in part devoted to mathematics. For many students, seeing mathematics explain physical phenomena is the most powerful motivation for its study. From that perspective, you might decide to choose the models you emphasize by the ideas that are exemplified in their analysis, that is, by selecting solely from the right-hand column of tables 3, 4, and 5.

3.1.5 My preferences

My own preferences for models to teach are:

- *scalar population models* (sections 2.1–2.2) because they are simple to derive, they are easy to understand without much background in the physical sciences, and they motivate all of the important first-order analytical, numerical, and graphical concepts,

- *the heat-flow model* (section 2.3) because it is easy to argue that it is plausible (although the underlying physics is subtle), it has an intu-

Ideas Used in Analyzing First-Order Models

Model	Section	Ideas used
Projectile motion		
Rock model: $v' = -g$	1.2	direction field, solution graph Euler's method, solution formula
	3.2	direction field, solution graph
Scalar population		
Simple: $P' = kP$	2.1	direction field, solution graph, solution formula
	3.1	Euler's, Heun's methods
	3.2	direction field, solution graph,
	4.1	general solution
	4.2–4.3	solution formula
Emigration: $P' = kP - E$	2.2.1	direction field, solution graph, steady state, stability
	3.2	direction field, solution graph, phase line
	3.3	steady state, stability
	4.1	general solution
	4.2, 4.4	solution formula
	10.4–10.5	Laplace transform solution
Logistic: $P' = aP - sP^2$	2.2.2; 3.2	direction field, solution graph, phase line
	3.3	linear stability analysis
	4.2	solution formula
	4.6	uniqueness
Heat flow		
$T' = -(Ak/cm)(T - T_{\text{out}})$	2.3	direction field, solution graph, steady state, stability, Euler's method
	3.1	Euler's, Heun's methods
	3.2	phase line
	3.3	steady state, stability
	4.1	general solution

Table 3: Analytical, numerical and graphical tools used in analyzing first-order models. The center column identifies the section in which the ideas listed in the right-hand column are used to analyze the given model. See tables 1 and 2 for the section in which each model is introduced.

Ideas Used in Analyzing Higher-Order Models

Model	Section	Ideas used
Multi-species population		
Predator-prey (2.24–2.25), p. 67	2.4.1	steady state, stability, Euler's method
Epidemic (SIR, SIRS) (2.29–2.30), p. 71	2.4.2	steady state, stability, phase plane
	7.2	nullclines, local linearization
Competition (2.35–2.36), p. 73	2.4.3	steady state, phase plane
Oscillators		
Spring-mass: $mx'' + px' + kx = F(t)$	5.1	solution formula, phase plane, Euler's method
	6.1	general solution
	6.3–6.5	solution formulas (homogeneous: undamped, overdamped, etc.
	6.6–6.7	solution formulas (nonhomogeneous: resonance, etc.)
Pendulum: $L\theta'' + g\sin\theta = 0$	5.2	linearization, solution formula
	6.1	steady state, linearization
	6.5, 6.8	solution formula (linear)
	6.8	steady state, linear stability
	7.1	phase plane
	7.2	nullclines, local linearization
	8.3	linear stability analysis, phase plane
van der Pol: $Li'' + \epsilon(i^2 - l)i' + i/C = 0$	7.3	limit cycle

Table 4: Analytical, numerical and graphical tools used in analyzing higher-order models. The center column identifies the section in which the ideas listed in the right-hand column are used to analyze the given model. See tables 1 and 2 for the section in which each model is introduced. (Continued in table 5.)

Ideas Used in Analyzing Higher-Order Models (cont'd)

Model	Section	Ideas used
Steady diffusion		
$-D(Ac')' + (AVc)' = 0$	9.2	solution formula
	9.3	numerical approximation (finite differences)
	11.2	solution formula
Time-dependent diffusion		
$\partial T/\partial t = \kappa \partial T^2/\partial x^2$	9.5	eigenfunction solution
	9.6	numerical approximation (method of lines), equilibria
	11.5	spatial eigenfunctions (Bessel)

Table 5: Continuation from table 4 of the list of analytical, numerical and graphical tools used in analyzing higher-order models. The center column identifies the section in which the ideas listed in the right-hand column are used to analyze the given model. See tables 1 and 2 for the section in which each model is introduced.

itively obvious stable steady state, and it seems more "real" to most students of science and engineering,

- *spring-mass and pendulum models* (sections 5.1–5.2) because these devices are easy to demonstrate in class, they introduce oscillatory phenomena, and they provide a complete foundation for most of the elementary two-dimensional ideas, particularly linearization and the phase plane,

- *SIRS* (or any other multiple species model from section 2.4) because for beginners the underlying science is simple and the phase plane is interesting,

- *diffusion models* (sections 9.1 and 9.4) because are an important class of boundary-value and initial-boundary-value problems for ordinary and partial differential equations, leading to important analytical and numerical ideas.

In class, I derive the population models in sections 2.1 and 2.2 (Malthus, emigration, logistic), argue for the plausibility of the heat-flow model (section 2.3), demonstrate a vertical spring-mass system and derive its governing equation (section 5.1), and derive one of the diffusion models in section 9.1. I use "Take my word for it", augmented by a little hand waving and a reading or a homework assignment, to introduce as needed the multiple species models (predator-prey, SIRS, competition) of section 2.4. The linear and nonlinear pendulum equations (section 5.2) enter with slightly more ceremony and homework emphasis but usually without a formal derivation.

My preferences among the mathematical ideas and methods are

- *graphical concepts* such as direction fields, phase planes, and sketching solution graphs from differential equations because they reinforce the rate of change concepts fundamental to calculus,

- *stability and linearization* because stability is an intuitive concept with natural analytic and graphical interpretations, its analysis often requires linearization (perhaps the most ubiquitous process, imperfections notwithstanding, in science and engineering), and linearization leads immediately back to the derivative and to Taylor's theorem.

- *elementary numerical methods* because they have natural graphical interpretations, they involve linearization, they call upon Taylor's theorem, and their sophisticated descendants are so important in practice,
- *elementary analytical methods* because they exercise manipulative skills, they provide a "plug 'n' chug" refuge for the student struggling with more difficult open-ended problems, and their limitations illustrate the importance of understanding. (To paraphrase Peter Hammer, "The purpose of differential equations is insight, not formulas (or numbers or graphs).")

3.1.6 Minimal modeling

For minimal attention to modeling in your course, use part of one lecture to touch the highlights of the derivation and analysis in chapter 1 of the rock model, $v' = -g$. Then follow the presentation in section 1.3 to use that simple example to motivate the elementary analytical, numerical, and graphical concepts to come.

Derive one of the population models, say $P' = kP'$ from section 2.1. Depend upon plausibility arguments and homework assignments like exercises 20 and 21, p. 79–80, of the chapter 2 chapter exercises for the emigration (subsection 2.2.1), logistic (subsection 2.2.2), and heat-flow equations (section 2.3).

Devote your other in-class derivation to the undamped spring-mass equation without forcing, $mx'' + kx = 0$ (section 5.1). Leave the linear and nonlinear pendulum equations ($L\theta'' + g\theta = 0$, etc.) to assigned reading of section 5.2.

Section 5.3, *RLC* Circuit (equations), and the subsequent analyses of these equations in subsections 6.5.3 (without forcing) and 6.7.2 (with forcing) can be omitted entirely.

Diffusion (section 9.1) is really too subtle to explore without a derivation in class, but the reality of life late in the semester of a minimal-modeling course is that you are likely to cover boundary-value problems fairly quickly. Introduce the boundary-value problems developed in examples 1 and 2, p. 432–433, in class and argue that the types of behavior they can support (e.g., a linear diffusion profile or circular symmetry) are reasonable.

The heat equation (section 9.4) ought to be derived in class if you are going to spend time developing the Fourier machinery of section 9.5 or the finite difference tools of section 9.6. The depth of coverage will depend upon your schedule and your students.

3.2 Numerical methods

Euler's method is introduced quickly in subsection 1.3.4, part of the survey in chapter 1, Prologue, of some of the major ideas in the text. Euler, Heun (RK2), and fourth-order Runge-Kutta are covered more completely in section 3.1, Numerical Methods. Finite difference methods for (ordinary) boundary-value problems are developed in section 9.3, Boundary-Value Problems: Numerical Methods, and the method of lines for the heat equation in section 9.6, Initial-Boundary-Value Problems: Numerical Methods.

There are two goals in these introductions, developing usable if unsophisticated numerical tools and building mathematical and computational intuition in anticipation of deeper study later in the curriculum. Of course, Euler, Heun, and RK4 are "baby" methods, intuitive building blocks for the adaptive initial-value methods available via DELab's access to Matlab's ode23, ode45, etc. The finite difference method and the method of lines are in a similar category, pedagogic rather than truly practical without more sophisticated enhancements.

3.2.1 Overview

The coverage of initial-value methods in section 3.1 is the deepest treatment of numerical ideas in the text;'the concepts of local error, global error, and numerical stability are introduced and analyzed at an appropriate level.

For the finite difference methods of section 9.3, accuracy is explored from two perspectives, the order of the derivative approximations being used and numerical experiments that study the variation in error with mesh size. No complete error analysis is given. The treatment of the methods of lines in section 9.6 is essentially purely formal.

DELab (see section 5 below) provides GUI-based access to the three fixed-step initial-value solvers Euler, Heun, and RK4, as well as Matlab's adaptive solvers ode23, ode45, ode113 and its stiff solvers ode23s and ode15s. It accommodates systems of arbitrary size, permitting easy solution of the systems of ordinary differential equations generated by the method of lines. As an alternative to direct use of matrix solver commands in the Matlab command window, it offers simple access to numerical solution of linear algebraic equations to support the use of finite differences.

3.2.2 Maximal numerical methods

Initial-value solvers After the quick introduction to Euler's method that is part of the survey in chapter 1, devote three or four days of lecture to a complete development of fixed-step initial-value solvers:

- *Day 1:* Subsection 3.1.1, four interpretations of the Euler method: as path following through the direction field (illustrated by DELAB (see section 5 below) or other interactive software), as the tangent to an exact solution curve, as an approximation to the derivative, as the first terms in a Taylor polynomial. Conclude with examples of accuracy, as in subsection 3.1.2.

- *Day 2:* Subsection 3.1.3, Better Methods: Use the geometric interpretation of Euler to motivate a better (second-order) method, Heun (RK2).

  Subsection 3.1.4, Global and Local Error: Use Taylor's theorem to analyze the local error in Euler and to demonstrate the superiority of Heun, one of several possible second-order Runge-Kutta methods.

  > I believe that the interpretation and analysis of Euler's method is one of the great opportunities to illustrate the real value of Taylor's theorem. Another is using Taylor as a tool for linearized analysis; e.g., example 48, section 6.8, p. 325. Taylor's theorem appears again in the discussion of finite difference approximations of derivatives in section 9.3, Boundary-Value Problems: Numerical Methods.

- *Day 3:* Subsection 3.1.5, An Even Better Method: Fourth-Order Runge-Kutta: Wave your hands to the effect that the Taylor series analysis that leads from Euler to RK2 can be continued (with a *lot* more algebra) to RK4, one version of which is stated on p. 99.

  Subsection 3.1.6, Numerical Stability, introduces a subtle but important idea. Certainly, numerical stability is one of the central pillars of a more mature understanding of initial-value solvers. An effective classroom approach is an interactive demonstration like that suggested in the MATLAB box on p. 101. An excursion into stiffness is irresistible at this point, but . . . .

To guide your assignment of exercises that support these lectures, note that the exercises in section 3.1 come in several varieties, including among others:

- *Exercising a method* and perhaps finding error, as in exercises 1–6 or 11–12
- *One-step approximations* that emphasize the nature of the underlying approximation, as in exercises 7–10.
- *Exploring variations in error with step size*, as in exercises 14–15, 25–26, 35–42, 45.
- *Using numerical approximations to analyze behavior or data*, as in exercises 16–23, 34.
- *Analytic and geometric interpretations of the methods*, as in exercises 27–32.
- *Efficiency*, as in exercises 13(b), 43–44.

For the sake of simplicity, all of the analysis in section 3.1 is presented for first-order scalar equations. Systems are regarded as a natural extension, as remarked on p. 103 of the text, and illustrated repeatedly, e.g., example 13, p. 18, in the quick introduction to Euler's method in chapter 1, Prologue.

Boundary-value solvers Coverage of finite-difference approximations in section 9.3, Boundary-Value Problems: Numerical Methods (for ordinary differential equations), and of the method of lines in Section 9.6, Initial-Boundary-Value Problems, will require about three days.

- *Day 1:* Subsections 9.3.1, Approximating Derivatives, and 9.3.2, Approximating Differential Equations: finite differences are introduced and used in differential equations.
- *Day 2:* Subsections 9.3.3, Incorporating Boundary Conditions, and 9.3.4, Derivative Boundary Conditions. (The latter can be omitted to save a little time if you are content with only Dirichlet boundary conditions.)

- *Day 3:* (after coverage of section 9.4, Time-dependent Diffusion, and (optionally) section 9.5, Fourier Methods) Section 9.6, Initial-Boundary-Value Problems, builds upon the finite difference ideas of section 9.3 to approximate the heat equation $T_t = \kappa T_{xx}$ by the method of lines.

  The analysis of the resulting systems of ordinary differential equations is in part analytic, using the linear systems ideas of section 8.2, Constant-Coefficient, Homogeneous Systems, to estimate rates of decay, for example, and in part numerical, calling upon the initial-value solvers Euler, Heun, and RK4 of section 3.1, Numerical Methods, to find numerical approximations. Such estimates are compared with the results of the Fourier analysis in section 9.5.

Projects Project 1 of chapter 9 introduces the shooting method, and project 3 of chapter 9 guides students through a simple finite difference approximation of the Laplacian on a square. Assignment of one or the other would add additional depth to the study of numerical methods.

3.2.3 Minimal numerical methods

A minimal treatment of numerical methods would quickly introduce Euler's method (subsection 1.3.4) as part of the survey presented in chapter 1, Prologue. A day devoted to the geometric interpretation of Euler's method (subsection 3.1.1) and a corresponding geometric motivation of Heun's method (subsection 3.1.3) could conclude with a statement of fourth-order Runge-Kutta (subsection 3.1.5). ("Intuitively, RK4 is better because it is averaging slopes at several points along the way from t to $t + \Delta t$.")

The range of exercises supporting initial-value solvers was described previously on p. 25.

Sections 9.3, Boundary-Value Problems: Numerical Methods, and 9.6, Initial-Boundary-Value Problems, can be omitted entirely without loss of continuity.

3.3 Analytical methods

The chapters devoted primarily to analytic methods are chapter 4 (first-order scalar), chapter 6 (second-order, mostly constant-coefficient), chapter 8 (systems, mostly constant-coefficient), chapter 10 (Laplace transforms), and chapter 11 (other second-order methods). Boundary-value problems are solved in section 9.2, and Fourier methods for the heat equation are developed in section 9.5, where second-order eigenvalue problems appear as well. Besides teaching the methods themselves, this coverage is organized to illustrate the process of generalization in mathematics by proceeding from simpler to more complex problems.

3.3.1 Overview

Chapter 4, Analytic Tools for One Dimensions, surveys some of the basic solution methods for scalar first-order equations. It treats characteristic equations, undetermined coefficients, and variation of parameters, all to set the stage for subsequent extensions of these methods.

The characteristic equation method is extended to second-order scalar equations in sections 6.3–6.4 (real and complex characteristic roots, respectively) and to systems of first-order equations in section 8.2.

The method of undetermined coefficients is extended to second-order equations in section 6.6. To alter the pattern of generalization, variation of parameters is extended to first-order systems in section 8.4, then specialized to second-order equations in 11.3.

The balance among the analytic methods favors the simpler constant-coefficient methods because they reveal relatively easily most of the features of the behavior of solutions of linear equations. Constant-coefficient equations arise in many models, and they are the natural outcome of linearized analyses. In addition, the inescapable patterns of generalization (e.g., extending characteristic equations from scalar to systems) and specialization (e.g., variation of parameters from systems to second-order) reveal a side of mathematics that too few students at this level appreciate.

> Since numerical (and graphical) methods are introduced very early in the text, you needn't fear accidentally encountering an equation that your students can't solve because you skipped a certain analytic method. Such equations always can be analyzed numerically or graphically (or solved on faith using the analytic (symbolic) tools in DELab).

The pattern of introduction of analytic methods is similar for first-order, second-order, and systems of equations. Basic definitions and concepts appear in the first section of the appropriate chapter (for scalar equations in section 4.1, for second-order equations in section 6.1 with the addition of linear independence tests in section 6.2, and for systems in section 8.1). Subsequent sections in each chapter develop the requisite solution machinery.

Systems of two first-order equations are the primary focus of chapter 8, Analytic Tools for Higher Dimensions. But the geometric (eigenvector) perspective used there extends easily and naturally when larger systems are encountered, as in the treatment in section 9.6 of the method of lines for the heat equation.

Second-order (ordinary) boundary-value problems are solved analytically in section 9.2, Boundary-value Problems: Analytic Tools. The heat equation is solved analytically using separation of variables and eigenfunction expansions in section 9.5.

Laplace transforms are the subject of chapter 10. They are motivated by the notion of sampling a solution function in search of rates of exponential growth or decay (or complex rates signifying oscillations). Although there are ample exercises in the usual manipulations, the spirit of the analysis goes beyond mere manipulation to reach two end points. One is accommodating piecewise continuous and impulsive forcing terms as in section 11.4, Ramps and Jumps, and 11.5, The Unit Impulse Function. The other goal is the sort of qualitative analysis exemplified by such exercises as 15–28 of the chapter 10 chapter exercises, p. 562–563.

Chapter 11 collects a number of familiar non-constant-coefficient solution methods for second-order equations. These can be covered in succession as ordered in the text or sampled from points earlier in the text as your preferences dictate.

For example, sections 11.1, Reduction of Order, and 11.2, Cauchy-Euler Equations, provide a general framework for solving homogeneous diffusion problems in a circle, of which there are examples in section 9.1, Diffusion Models. So 11.1 and 11.2 could immediately follow 9.1 if desired.

Moreover, solving nonhomogeneous diffusion equations in a circle would require section 11.3, Variation of Parameters: Second-order Equations. Hence, that section could join 11.1 and 11.2, immediately following 9.1.

In a completely different way, the idea of specializing from first-order systems back to a second-order equation could motivate a brief study of 11.3, Variation of Parameters: Second-order Equations, immediately following 8.4, Nonhomogeneous Systems: Variation of Parameters.

Similar connections link power series methods and the eigenfunction problems that arise from applying the ideas of section 9.5, Fourier Methods, to the heat equation in a circle. Section 11.4, Power Series Methods, and section

11.5, Regular Singular Points, could be scheduled just after 9.5 to find those eigenfunctions and to characterize their behavior near the origin.

To nurture mathematical maturity, some familiar analytic methods are left to students to develop. Among them are:

- *Integrating factor for linear, first-order equations* in projects 1 and 2 of chapter 4, p. 200–201,
- *Constant-coefficient methods for third and higher order* in exercises as 19–24 of the chapter 6 chapter exercises, p. 330,
- *Undetermined coefficients for first-order systems* in exercise 24 of section 8.4, p. 423,
- *Laplace transforms for systems* in exercises as 29–36 of the chapter 10 chapter exercises, p. 563.

3.3.2 Maximal analytic solutions

Complete coverage of analytic solution methods could begin with a quick look at separation of variables in subsection 1.3.3, then work methodically through the first five sections of chapter 4: 4.1, Basic Definitions; 4.2, Separation of Variables; 4.3, Characteristic Equations; 4.4, Undetermined Coefficients; and 4.5, Variations of Parameters. Section 4.6, Uniqueness and Existence, might reasonably be part of such a thorough treatment.

> Uniqueness is treated first in a separate subsection within section 4.6 so that it can be covered without considering existence, if necessary. The machinery of uniqueness is a bit easier than that of existence, and its graphical consequences often seem more significant to students than existence, profound differences notwithstanding.

A correspondingly thorough treatment of second-order equations would cover sections 6.1, Basic Definitions; 6.2, Testing Linear Independence; 6.3, Characteristic Equations: Real Roots; 6.4, Characteristic Equations: Complex Roots; and 6.6, Undetermined Coefficients.

Sections 6.5, Analyzing Models without Forcing, and 6.7, Analyzing Models with Forcing, contain examples of the application and utility of characteristic equations and undetermined coefficients (e.g., to the discovery of resonance). In the context of a thorough study of analytic methods, section

6.8, Linear versus Nonlinear, shows how constant-coefficient linear equations, those for which students have solution methods, arise naturally in the course of a linear stability analysis.

The four sections of chapter 8 emphasize constant-coefficient systems: 8.1, Basic Definitions: Systems; 8.2, Constant-coefficient Homogeneous Systems; 8.3, Connections with the Phase Plane; 8.4, Nonhomogeneous Systems: Variation of Parameters. Section 8.3 plays a role parallel to that of section 6.8, Linear versus Nonlinear, for second-order equations. It connects nonlinear and linear systems through a linearized analysis in the phase plane, illustrating the importance of the eigenvalue-eigenvector understanding of constant-coefficient, homogeneous systems. Of course, the phase plane perspective of section 8.3 is much richer than the elementary analytic view taken in section 6.8.

Note that the vector-matrix view of systems is introduced gradually in section 8.1, Basic Definitions: Systems, without assuming prior instruction in linear algebra. Additional review of matrix concepts is provided in appendix section A.5, material you may wish to incorporate if your students are particularly uncertain about these ideas.

> I find that MATLAB's eigshow is a powerful classroom demonstration in two respects. It illustrates the notion of matrix multiplication as a vector input-output operation, and it nails down a geometric picture of the meaning of a real eigenvalue. A few minutes of eigshow is worth an hour of chalk and hand waving! Type help eigshow in the MATLAB command window or access eigshow from the Analytic tools menu bar selection in DELAB.

Section 9.2 uses the second-order solution tools of chapter 6 to solve boundary-value problems, which are the diffusion models introduced in section 9.1.

A syllabus that intended to cover every analytic method remaining in the text would then proceed sequentially from 9.2, Boundary-value Problems, through section 9.4, Time-dependent Diffusion (to derive the heat equation), section 9.5, Fourier Methods, chapter 10, The Laplace Transform, and chapter 11, More Analytic Methods for Two Dimensions.

If partial differential equations and Laplace transforms were not of interest, one could move directly from the diffusion models and boundary-value problems of sections 9.1–9.2 to sections 11.1–11.3 to cover reduction of order, Cauchy-Euler equations, and variation of parameters, solution tools that can handle such non-constant-coefficient equations as models of diffusion in a circle.

A more ambitious tour of analytic methods would work through section 9.5, Fourier Methods, a long section that should be covered one subsection at a time. It could then turn to sections 11.4 and 11.5 for series solution methods, motivated in part by the eigenfunction equation that arises from separating the heat equation in a circle. Of course, the five sections of chapter 10, The Laplace Transform, could be covered last to complete a full tour in a different order.

In summary, a complete tour of all analytic methods would visit the following sections:

- 1.3: introduction to solution concepts
- 4.1–4.6: first-order initial-value problems
- 6.1–6.4, 6.6, 6.8: second-order initial-value problems
- 8.1–8.4: first-order systems
- 9.2: second-order (ordinary) boundary-value problems
- 9.5: Fourier methods for the heat equation
- 10.1–10.5: Laplace transform
- 11.1–11.6: additional second-order methods, including series

3.3.3 Minimal analytic solutions

A minimal treatment of analytic solution methods requires a glance at separation of variables (section 1.3.3), basic definitions and techniques for first- and second-order linear, constant-coefficient scalar equations (4.1, 4.3–4.4, 6.1–6.4, 6.6), and constant-coefficient, homogeneous, linear systems (8.1–8.2). Extrapolate between this bare minimum and the full coverage of described in the previous subsection to incorporate additional material you believe appropriate.

3.4 Graphs and phase diagrams

Graphical ideas are introduced early (e.g., the direction field of figure 1.2, p. 4) and used regularly. Sketching solution graphs of first-order scalar equations (e.g., section 1.2, A Modeling Example; section 2.1, Simple Population Models; section 2.2, Emigration and Competition, etc.) reinforces slope and concavity ideas from calculus, providing an obvious connection with earlier course work that many students find reassuring.

After regular but informal use of direction fields and solution graphs to analyze the projectile model of chapter 1 and the various population models of chapter 2, these ideas are consolidated in section 3.2, Direction Fields and Phase Lines. They are further reinforced in section 3.3, Steady States, Stability, and Linearization.

The phase plane first appears in the analysis of the population models derived in section 2.4, Multiple Species. It is introduced more formally in chapter 7, primarily through the nonlinear pendulum equation, and tied to the geometry of the solutions of constant-coefficient, homogeneous, linear systems in section 8.3, Connections with the Phase Plane.

A maximal treatment of graphical ideas would start with section 1.2, then include all of sections 3.2, 3.3, 7.1–7.3, and 8.3.

A minimal treatment might omit subsection 3.2.3, Phase Lines, tread lightly on the graphical interpretations of stability in section 3.3, and omit section 7.2, Nullclines and Local Linearization, and section 7.3, Limit Cycles and Stability. Conceivably, section 8.3, Connections with the Phase Plane, could be omitted as well, but you would be depriving your students of the two-dimensional punch line that connects analytic and geometric ideas. Without it, differential equations will seem a long shaggy dog story about tricks and special methods.

These graphical results—sketching solution plots, direction fields, phase diagrams, etc.—can obtained through the Graphical tools menu bar selection in DELab.

3.5 Partial differential equations

The primary partial differential equation treated in this text is the heat equation $T_t = \kappa T_{xx}$ and its relatives in other geometries. It is derived in section 9.4, then solved by separation of variables and Fourier series in section 9.5 and by the method of lines in section 9.6. Separating this equation in a circle provides a key example in section 11.5, Regular Singular Points.

The equilibrium solutions of this equation are defined by ordinary boundary-value problems, providing a direct tie to the material of section 9.2, Boundary-value Problems: Analytic Tools, and section 9.3, Boundary-value Problems: Numerical Methods. Furthermore, analyzing rates of approach to equilibrium makes use of the analytic methods of section 9.5, Fourier Methods, and the numerical methods of section 9.6, Initial-Boundary-value Problems: Numerical Methods. Such qualitative analyses are less common in the usual elementary study of partial differential equations, but they are consistent with the spirit of model-analyze-interpret that runs through the text.

4 Section by section emphasis

Table 6 identifies the primary emphasis of each section relative to

- derivation or analysis of a model,
- development and application of one of the three categories of ideas,
  - analytical,
  - numerical,
  - graphical.

5 Software

This text is supported by DELab, a graphical user interface to Matlab's powerful suite of tools. Users can either enter directly or select from a list a differential equation or a system, then choose to find an analytic (symbolic) solution, a numerical approximation, or a graphical display. DELab has been designed to meet two goals: providing easy access to Matlab without introducing unnecessary limitations and fostering effective use of Matlab's full capabilities.

DELab requires no knowledge of Matlab, and use of its interface is largely self-evident, though on-line help is available. However, users who are familiar with Matlab can return most symbolic, numerical, or graphical

Primary emphasis by section

Models and their analysis	**Analytical tools**	**Numerical tools**	**Graphical tools**
Chapter 1, Prologue, samples all four areas.			
2.1–2.4			
	3.3	3.1	3.2
	4.1–4.6		
5.1–5.3			
6.6, 6.7	6.1–6.4, 6.8		
			7.1–7.3
	8.1–8.2, 8.4		8.3
9.1, 9.4	9.2, 9.5	9.3, 9.6	
	10.1–10.5		
	11.1–11.6		

Table 6: The primary emphasis of each section

results from DELab to the Matlab command window for further study or otherwise exploit Matlab's vast capabilities to perform a deeper analysis.

For example, a symbolic solution found through DELab can be placed in the Matlab command window as a new symbolic variable so that the user can find formulas for inflection points. Or it could be put into an inline function so that the user could draw a surface plot illustrating variation in behavior with time and initial value.

DELab requires Matlab 5.2 and its symbolic toolbox or the student edition of Matlab. More information about DELab as well as the code itself is available from `www.wpi.edu/~pwdavis/DELab`

6 Instructor aids

A solutions manual is available directly from Prentice Hall; contact Gale Epps, `Gale_Epps@prenhall.com`, 1-201-236-7405. Additional exercises and examples are posted regularly to `www.wpi.edu/~pwdavis/ModelingWithMatlab`.

Other comments, suggestions, and ideas are welcome. Please send them to me at `pwdavis@wpi.edu`, with my thanks in advance.