MA 1024 – Lagrange Multipliers for Inequality Constraints

Here are some suggestions and additional details for using Lagrange multipliers for problems with *inequality* constraints.

Statements of Lagrange multiplier formulations with multiple *equality* constraints appear on p. 978-979, of Edwards and Penney's *Calculus Early Transcendentals*, 7th ed. Refer to them. Note the condition that the gradients of the constraints (e.g., ∇g and ∇h in Theorem 2, p. 978) must be nonzero and nonparallel.

Here's an example with *inequality* constraints: **find the minimum of** $f(x) = x^2$ **for** $1 \leq x \leq 2$[1].

Written separately, the inequality constraints are $x-1 \geq 0$ and $2-x \geq 0$. To convert them to *equality* constraints, **introduce two new variables** s **and** t **and corresponding equality constraints:**

$$\begin{aligned} g_1(x,s,t) &= x - 1 - s^2 = 0 \\ g_2(x,s,t) &= 2 - x - t^2 = 0 \end{aligned}$$

Squaring the new variables insures that these terms are non-negative, thereby capturing the inequality constraints. The variables s and t are called *slack variables* because they take up the slack in the inequalities.

The revised problem is: **minimize** $f(x) = x^2$ **subject to** $g_1(x,s,t) = x - 1 - s^2 = 0$ **and** $g_2(x,s,t) = 2 - x - t^2 = 0$**.**

The **Lagrange multiplier formulation** is: solve

$$\begin{aligned} g_1(x,s,t) &= x - 1 - s^2 = 0 \\ g_2(x,s,t) &= 2 - x - t^2 = 0 \\ \nabla f(x) &= \lambda_1 \nabla g_1(x,s,t) + \lambda_2 \nabla g_2(x,s,t) \end{aligned}$$

The gradient operates on the three variables (x,s,t); i.e., $\nabla = \langle \partial_x, \partial_s, \partial_t \rangle$. Hence,

$$\begin{aligned} \nabla f &= \langle 2x, 0, 0 \rangle \\ \nabla g_1 &= \langle 1, -2s, 0 \rangle \\ \nabla g_2 &= \langle -1, 0, -2t \rangle \end{aligned}$$

[1] Of course, the minimum occurs at $x = 1$.

Hence, the **five Lagrange multiplier equations** are

$$\begin{aligned} x - 1 - s^2 &= 0 & (1)\\ 2 - x - t^2 &= 0 & (2)\\ 2x &= \lambda_1 - \lambda_2 & (3)\\ 0 &= -2s\lambda_1 & (4)\\ 0 &= -2t\lambda_2 & (5) \end{aligned}$$

There are two possibilities with each inequality constraint, *active* – up against its limit – or *inactive*, a strict inequality. If the constraint is active, the corresponding slack variable is zero; e.g., if $x - 1 = 0$, then $s = 0$. The inequality constraint is actually functioning like an equality, and its Lagrange multiplier is nonzero. If the inequality constraint is inactive, it really doesn't matter; its Lagrange multiplier is zero.

Equations (4) and (5) offer these alternatives:

From (4): either $s \neq 0$ and $\lambda_1 = 0$, in which case $x = 1 + s^2 > 1$, *or* $s = 0$ and $x = 1$, from (1).

From (5): either $t \neq 0$ and $\lambda_2 = 0$, in which case $x = 2 - t^2 < 2$, *or* $t = 0$ and $x = 2$, from (2).

Hence, according to equations (1), (2), (4) and (5), just one of the following is possible:

Case I: $x = 1$.

Case II: $x = 2$.

Case III: $1 < x < 2$ and $\lambda_1 = \lambda_2 = 0$.

In Case III, equation (3) forces $x = (\lambda_1 - \lambda_2)/2 = 0$, violating $1 < x < 2$. So III is impossible.

Which of I and II gives the lower value to f? Of course, it's Case I; $x = 1$ gives $f(x) = x^2$ its minimum value on $1 \leq x \leq 2$.

Checking points to eliminate impossibilities and to locate the actual extremum is typical of Lagrange multipliers.