PH1140 D09 Homework 5 Solution

1.[130] Membrane Wave. The wave function given by

$$\zeta(x,z,t) = A\cos(k_x x + k_z z - \omega t + \phi)$$

specifies the transverse displacement of a plane wave moving along a thin elastic membrane of mass density $\sigma = 0.181\ kg/m^2$, where

$$A = 2.7\ cm, \qquad \omega = 22\pi\frac{rad}{s}, \qquad k_x = 30\frac{rad}{m}, \qquad k_z = 40\frac{rad}{m}, \qquad \varphi = \frac{\pi}{12}$$

a) [20] Write the wave vector in rectangular and polar form. What is the direction of the wave?

The wave vector is

$$\vec{\boldsymbol{k}} = k_x\hat{\boldsymbol{x}} + k_z\hat{\boldsymbol{z}} = 30\frac{rad}{m}\hat{\boldsymbol{x}} + 40\frac{rad}{m}\hat{\boldsymbol{z}} = 50\frac{rad}{m}\measuredangle 0.927\ rad$$

The direction is $\theta = 0.927\ rad$.

b) [30] What are the wave length, wave number and wave speed?

The wave number is $k = \sqrt{k_x{}^2 + k_z{}^2} = 50\ rad/m$.

The wave length is

$$\lambda = \frac{2\pi}{k} = \frac{\pi}{25}m = 0.04\pi\ m = 0.1257\ m.$$

The wave speed is $v = \omega/k = 1.382\ m/s$.

c) [20] A wave front is a line of constant phase defined by $\theta = k_x x + k_z z - \omega t + \phi = constant$. At the time $t = (1/33)\ s$, graph in the $\{x,z\}$ plane two consecutive wave fronts defined respectively by the two constant phases $\theta = 0$ and $\theta = 2\pi$. Indicate on the graph the direction of the wave and the wave length. Calculate the distance from the $\{x,z\}$ origin to the nearest of the two wave fronts, and indicate this distance on the graph.

The wave front defined by $\theta = 0$ and $t = (1/33)\ s$ is the line in the $\{x,z\}$ plane given by

$$\theta = 0 = k_x x + k_z z - \omega t + \phi = 30x + 40z - \frac{2}{3}\pi + \frac{1}{12}\pi = 30x + 40z - \frac{7}{12}\pi$$

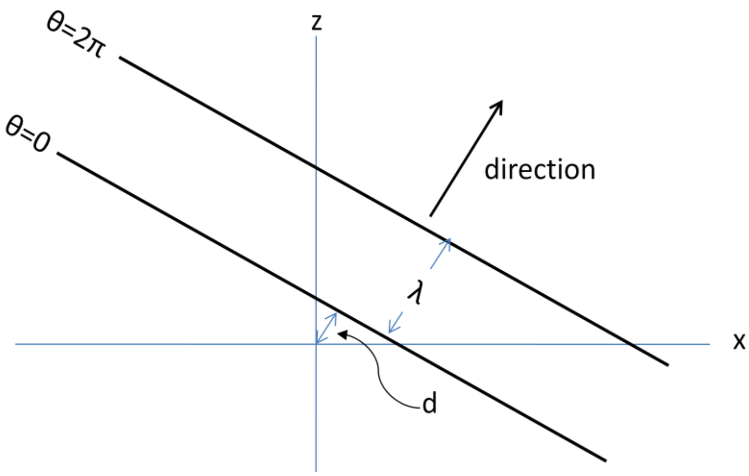
which may be written in the slope-intercept form $z = mx + b$

$$z = -\frac{3}{4}x - \frac{7}{480}\pi = -\frac{3}{4}x - 0.0146\pi = -\frac{3}{4}x - 0.0458.$$

Similarly the wave front defined by $\theta = 2\pi$ and $t = (1/33)\ s$ is the line in the $\{x,z\}$ plane given by

$$z = -\frac{3}{4}x - \frac{31}{480}\pi = -\frac{3}{4}x - 0.0646\pi = -\frac{3}{4}x - 0.203.$$

These lines (wave fronts) are shown in the figure below. The waves move in the indicated direction, and the perpendicular distance between the wave fronts is the wave length λ.



d) [10] Indicate on the graph the direction of the waves and the wavelength. See figure.

e) [20] Calculate the distance from the $\{x, z\}$ origin to the nearest of the two wave fronts, and indicate this distance on the graph.

To find the distance d between the origin and the first wave front we ask "what is the phase of the wave at the origin at time $t = (1/33)\ s$"? Since the coordinates x and z are both zero at the origin, the phase is

$$\theta_{origin} = k_x x + k_z z - \omega t + \phi = -\omega t + \phi = -\frac{7}{12}\pi\, rad.$$

Since the first wave front is at phase $\theta = 0$, the phase difference between the origin and the first wave front is

$$\Delta\theta = \theta - \theta_{origin} = \frac{7}{12}\pi\, rad.$$

The wave number k tells us the phase change in radians per distance traveled by a wave front, thus

$$k = \frac{2\pi}{\lambda} = \frac{\Delta\theta}{d}$$

Solving for d we have $d = 0.0367\ m.$

f) [20] What are the maximum transverse displacement, speed and acceleration of a point on the membrane?

The maximum transverse displacement is $\zeta_{max} = A = 2.7\ cm.$

The maximum transverse speed is $v_{\zeta,max} = \omega A = 1.867\ m/s.$

The maximum transverse acceleration is $a_{\zeta,max} = \omega^2 A = 129.0\ m/s^2.$

g) [10] What is the tension per unit length in the membrane?

The tension per unit length is

$$F = v^2\sigma = 0.346\ N/m.$$

2. [80] Ocean Waves. Tabulate the wave speed $v(k,h)$ and frequency $\omega(k,h)$ of ocean waves for the wave lengths and ocean depths given below. Suggestion: you can work this problem quickly by implementing the formulas given in lecture (also in Study Guide 3) in a Microsoft Excel spreadsheet, where you can utilize the built-in hyperbolic trigonometric functions. In the formula for displacement, use $A = 1\ m^2/s$.

$\lambda\ (m)$	$h\ (m)$
1	0.1, 1.0, 10, 100
1000	10, 100, 1000, 10000

The results are tabulated below. Note that when the depth is equal to or greater than the wave length (relatively deep water), the speed and frequency are fairly constant. When the depth is much smaller than the wave length (relatively shallow water), the frequency and speed are diminished.

h (m)	λ (m)	k (rad/m)	ω (rad/s)	v (m/s)
0.01	1	6.28	1.97	0.31
0.1	1	6.28	5.86	0.93
1	1	6.28	7.85	1.25
10	1	6.28	7.85	1.25
100	1	6.28	7.85	1.25
10	1000	0.01	0.062	9.89
100	1000	0.01	0.185	29.47
1000	1000	0.01	0.248	39.49
10000	1000	0.01	0.248	39.49

3. [80] Ocean Waves. A seismic event occurs beneath the ocean at a place where the ocean depth is $h_{source} = 3000\ m$. The event creates a train of traveling ocean plane waves of amplitude $\eta_{max}(h_{source}) = 1.0\ m$, at a wave length of $\lambda_{source} = 6000\ m$. When wave train reaches a small boat located where the depth is $h_{boat} = 1000\ m$, the wave length has shortened to $\lambda_{boat} = 5070\ m$.

a) [10] The waves arrive at the boat with frequency f_{boat}. How is this frequency related to the frequency f_{source} of the waves at the source?

The waves in the wave train are identical, and therefore travel from source to boat in the same amount of time Δt. Two consecutive waves leave the source at times t_1 and t_2. In terms of these times, the period is $T = t_2 - t_1$. The waves arrive at the boat at times $t_1' = t_1 + \Delta t$ and $t_2' = t_2 + \Delta t$. The difference in arrival times is $t_2' - t_1' = (t_2 + \Delta t) - (t_1 + \Delta t) = T$. The period is the same at the source and the boat, therefore the frequencies are equal, $f_{source} = f_{boat}$.

b) [30] Determine the frequency f_s, wave number k_s and wave speed v_s at the source.

The wave number is

$$k_s = \frac{2\pi}{\lambda_s} = 0.001047 \frac{rad}{m}.$$

The frequency is

$$f_s = \frac{\omega_s}{2\pi} = \frac{\sqrt{gk_s \tanh(k_s h_s)}}{2\pi} = 0.01609\ Hz\,.$$

The wave speed is

$$v_s = \sqrt{\frac{g}{k_s}\tanh(k_s h_s)} = 96.6\frac{m}{s} = 216\ mph\,!\,.$$

c) [30] Determine the frequency f_b, wave number k_b and wave speed v_b at the boat.

The wave number is

$$k_b = \frac{2\pi}{\lambda_b} = 0.001239 \frac{rad}{m}.$$

From part (a), the frequency is unchanged

$$f_b = f_s = 0.01609\ Hz\,.$$

The wave speed is

$$v_b = \sqrt{\frac{g}{k_b}\tanh(k_b h_b)} = 81.8\frac{m}{s}.$$

d) [10] Determine the amplitude $\eta_{max}\,(h_{boat})$ of the waves at the boat.

Skip part (d) !

4. [20] Show that the two traveling waves $\zeta(x,z,t) = A\cos(k_x x + k_z z - \omega t + \phi)$ and $\zeta(x,z,t) = A\cos(k_x x + k_z z + \omega t + \phi)$ combine to produce a standing wave. Hint: apply trigonometric identities involving the sum of two angles, and treat each of the wave phases as a sum of two angles.

Let $a = k_x x + k_z z + \phi$ and $b = \omega t$ in the identities

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,, \qquad \cos(a-b) = \cos a \cos b + \sin a \sin b\,.$$

Adding the identities

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b\,,$$

substituting for a and b, and multiplying through by amplitude A leads to

$$A\cos(k_x x + k_z z - \omega t + \phi) + A\cos(k_x x + k_z z + \omega t + \phi) = 2A\cos(k_x x + k_z z + \phi)\cos(\omega t)\,.$$

The expression on the right is a standing wave, the product of two sinusoids, one a function of space coordinates, the other a function of the time coordinate.

5. [80] Standing Waves and Modes on Rectangular and Circular Membranes.

Use the two web applets http://www.falstad.com/circosc/ and http://www.falstad.com/membrane/ to study the modes of rectangular and circular membranes. Read the Full Directions to learn how to adjust magnitude and phase.

For the rectangular membrane, you use rectangular coordinates x and y to indicate the positions of nodes and anti-nodes. Also, use O's and X's to distinguish between crests and troughs.

For the circular membrane, you use polar coordinates r and φ to indicate the positions of nodes and anti-nodes. Use circles and dotted circles to distinguish between radial crests and troughs. Use O's and X's to distinguish between angular crests and troughs.

a) [20] For $n_x = 1$ and $n_y = 2$, sketch the mode of the rectangular membrane.

b) [20] For $n_x = 2$ and $n_y = 2$, sketch the mode of the rectangular membrane.

c) [20] For $n_r = n = 2$ and $n_\varphi = m = 0$, sketch the mode of the circular membrane.

d) [20] For $n_r = n = 1$ and $n_\varphi = m = 1b$, sketch the mode of the circular membrane.