

PH1140 D09 Homework 4 Solution [390 points]

The numbers in brackets [] are the grade point values.

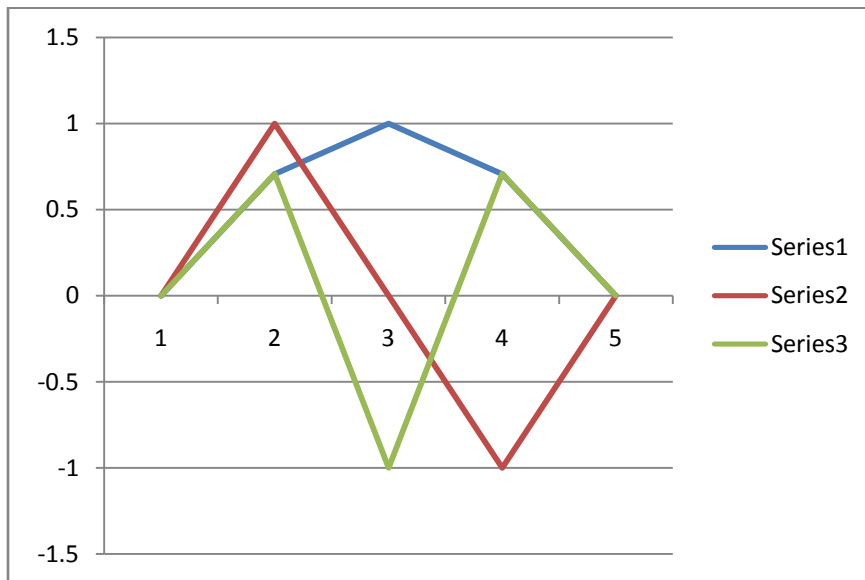
1. [50] Working with the A_{pn} and $y_{pn}(t)$ from Lecture (also see Study Guide 2). For a coupled system of $N = 3$ carts:

a) [30] Determine the transverse amplitudes A_{pn} and the displacements $y_{pn}(t)$ for carts $p = 1, 2, 3$, in each of the modes $n = 1, 2, 3$. Also determine the frequencies ω_n , wave numbers k_n , wave lengths λ_n and wave speeds v_n for each of the modes.

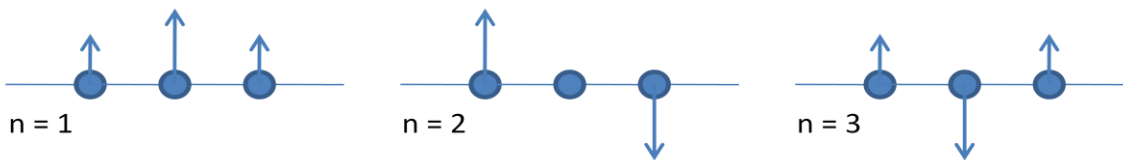
Table of A_{np} $C_n = 1, \omega_0 = 1, \delta_n = 0$ $N = 3$

| n = | p = | | | k_n | λ_n | ω_n | v_n |
|-----|-------|--------|--------|-------|-------------|------------|-------|
| | 1 | 2 | 3 | | | | |
| | 0 | 0 | 0 | | | | |
| 1 | 0.707 | 1.000 | 0.707 | 0.785 | 8.0 | 0.765 | 0.974 |
| 2 | 1.000 | 0.000 | -1.000 | 1.571 | 4.0 | 1.414 | 0.900 |
| 3 | 0.707 | -1.000 | 0.707 | 2.356 | 2.7 | 1.848 | 0.784 |

b) [10] Sketch the modes with carts fully displaced.

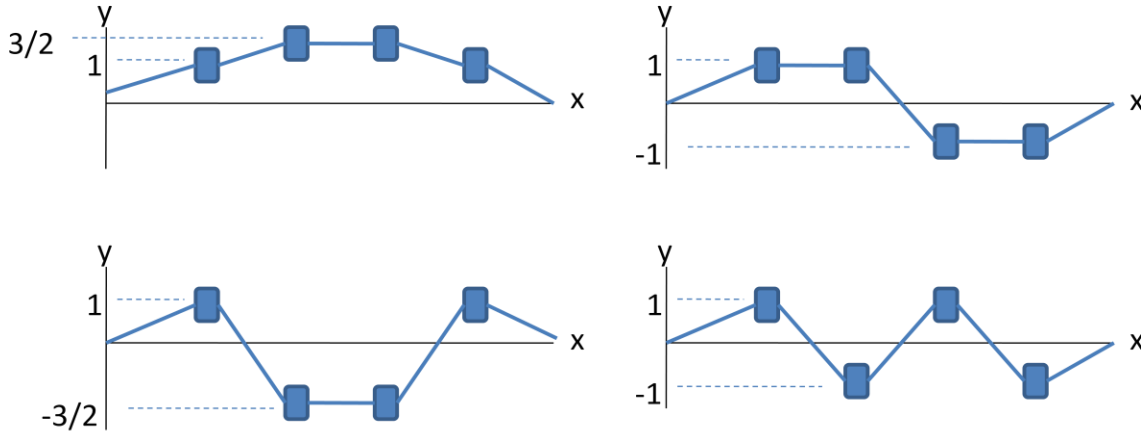


c) [10] Sketch the modes with carts in their equilibrium positions, and indicate the cart velocities with arrows.

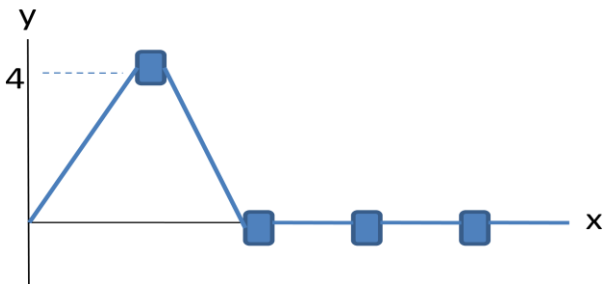


2. [30] Transverse Oscillations of Four Carts. From your experience in lab, you know that four carts on elastic string have four transverse normal oscillation modes. Suppose that the four modes can be created, respectively, by the four sets of initial displacements shown below. For example, reading the mode one diagram (first diagram), the cart displacements are

$$y_1 = 1, \quad y_2 = \frac{3}{2}, \quad y_3 = \frac{3}{2}, \quad y_4 = 1.$$



a) [10] Add the displacements of all four modes together to show that the combination of the four sets of displacements is equivalent to the set of displacements shown below.



Solution. Adding the displacements for cart 1:

$$y_1 = 1 + 1 + 1 + 1 = 4.$$

Doing the same for carts 2, 3 and 4:

$$y_2 = \frac{3}{2} - \frac{3}{2} + 1 - 1 = 0, \quad y_3 = \frac{3}{2} - \frac{3}{2} - 1 + 1 = 0, \quad y_4 = 1 + 1 - 1 - 1 = 0.$$

From part (a), it is clear that the initial displacement of a single cart is equivalent to a sum of initial displacements, and “contains” a little bit of each of the four normal modes. In lab, you observed how, when a single cart is given an initial displacement and released, its motion is carried to the second cart, then the third, and finally the fourth, forming wave motion. This wave motion is a superposition of the normal modes, and is therefore a superposition of traveling waves involving four different wave speeds.

b) [20] Starting with the four carts in their equilibrium positions, you displace the first cart and release it to initiate transverse motion. Assume uniform cart spacing l such that the string length is given by

$L = (N + 1)l$, where $N = 4$. In terms of the string tension T , the cart spacing l , and the cart mass m , how long does it take for the fourth cart to “feel” the motion?

Solution: From the formulas for the angular frequency, wave number and wave speed

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right], \quad \omega_0^2 = T/ml, \quad k_n = n\pi/L, \quad v_n = \frac{\omega_n}{k_n}$$

we calculate the speeds of the traveling waves of the four modes

$$v_1 = 0.984\sqrt{Tl/m}, \quad v_2 = 0.935\sqrt{Tl/m}, \quad v_3 = 0.858\sqrt{Tl/m}, \quad v_4 = 0.757\sqrt{Tl/m}.$$

The first mode’s traveling waves are the fastest, so the motion of cart 1 will reach cart 4 in a time

$$t = \frac{d}{v_1} = \frac{3l}{v_1} = 3.05 \sqrt{\frac{ml}{T}}.$$

3. [30] Y&F Exercise 15.1. Sound waves have many characteristics in common with waves on a string. A sound wave, like a wave on a string, has a period T , frequency f , wave length λ , wave number k and wave speed v . The following familiar formulas also apply to sound waves:

$$T = \frac{1}{f}, \quad \omega = 2\pi f, \quad k = \frac{2\pi}{\lambda}, \quad v = \frac{\omega}{k} = \lambda/T.$$

a) The wavelength is

$$\lambda = \frac{v}{f} = \frac{344 \text{ m/s}}{784 \text{ Hz}} = 0.439 \text{ m}.$$

The period is

$$T = \frac{1}{f} = 1.28 \text{ ms}.$$

b) One octave higher means frequency is doubled and wave length is halved

$$\lambda = 0.219 \text{ m}.$$

c) What are the wave numbers of the waves in parts (a) and (b)? From $k = 2\pi/\lambda$ the wave numbers are

$$(a) \quad k = 2\pi/\lambda = 14.3 \text{ rad/m}$$

$$(b) \quad k = 2\pi/\lambda = 28.7 \text{ rad/m}$$

4. [30] Y&F Exercise 15.5. Light behaves like a wave, having the same characteristics given above, namely T , f , ω , k and v .

a) Frequency is given in terms of speed and wave length by $f = c/\lambda$.

$$f_{\text{violet}} = \frac{c}{\lambda_{\text{violet}}} = \frac{3 \times 10^8 \text{ m/s}}{400 \text{ nm}} = 7.5 \times 10^{14} \text{ Hz}, \quad T_{\text{violet}} = 1.33 \times 10^{-15} \text{ s}$$

$$f_{red} = \frac{c}{\lambda_{red}} = \frac{3 \times 10^8 \text{ m/s}}{700 \text{ nm}} = 4.29 \times 10^{14} \text{ Hz}, \quad T_{red} = 2.33 \times 10^{-15} \text{ s}$$

b) Stop watches typically measure hundredths of a second, far too coarse to measure the period of visible light.

c) The wave numbers of the waves in parts (a) and (b) are

$$(a) \quad k_{violet} = 2\pi/\lambda_{violet} = 15.71 \times 10^6 \text{ rad/m}$$

$$(b) \quad k_{red} = 2\pi/\lambda_{red} = 8.98 \times 10^6 \text{ rad/m}$$

5. [40] Y&F Exercise 15.12.

a) Equation 15.3 may be rewritten by taking the factor $1/v$ out of the parenthesis and using $v = \lambda f$

$$y(x, t) = A \cos 2\pi f \left(\frac{x}{v} - t \right) = A \cos 2\pi \frac{f}{v} (x - vt) = A \cos \frac{2\pi}{\lambda} (x - vt)$$

b) Transverse velocity is given by

$$v_y = \dot{y} = \frac{dy}{dt} = \frac{2\pi v}{\lambda} A \sin \frac{2\pi}{\lambda} (x - vt)$$

c) The maximum transverse velocity occurs when

$$\sin \frac{2\pi}{\lambda} (x - vt) = 1$$

giving

$$v_{y,max} = \frac{2\pi A}{\lambda} v = A k v$$

Then if

$$\begin{aligned} \frac{2\pi A}{\lambda} &= 1, & v_{y,max} &= v \\ \frac{2\pi A}{\lambda} &> 1, & v_{y,max} &> v \\ \frac{2\pi A}{\lambda} &< 1, & v_{y,max} &< v \end{aligned}$$

d) Also show that the forms of eq. 15.3 in part (a) are equivalent to $y(x, t) = A \cos(kx - \omega t)$.

First step: take the factor $2\pi f$ inside the parenthesis; second step: substitute $v = \lambda f$ and $f = 1/T$; third step: recognize that $k = 2\pi/\lambda$ and $\omega = 2\pi/T$.

$$y(x, t) = A \cos 2\pi f \left(\frac{x}{v} - t \right) = A \cos \left(\frac{2\pi f}{v} x - 2\pi f t \right) = A \cos \left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t \right) = A \cos(kx - \omega t).$$

6. [20] Y&F Exercise 15.14.

Solve $v = \sqrt{F/\mu}$ for the tension F to obtain $F = \mu v^2$.

Substituting $v = f\lambda$ and $\mu = m/L$, and inserting numerical values

$$F = \mu v^2 = \frac{m}{L} (f\lambda)^2 = \frac{0.12 \text{ kg}}{2.5 \text{ m}} (40 \text{ Hz})^2 (0.75 \text{ m})^2 = 43.2 \text{ N}$$

7. [70] Y&F Exercise 15.24.

a) The amplitude is

$$A = 2.3 \text{ mm.}$$

b) The frequency is

$$f = \frac{\omega}{2\pi} = \frac{742}{\pi} = 118 \text{ Hz.}$$

c) The wave length is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{6.98} = 0.90 \text{ m.}$$

d) The wave speed is

$$v = \frac{\omega}{k} = 106 \frac{\text{m}}{\text{s}}.$$

e) The direction of travel is found from the motion of a point of constant phase (simplest is to set phase equal to zero)

$$kx + \omega t = 0$$

$$x = -\frac{\omega}{k} t = -vt$$

The speed is negative, so the wave moves in the $-x$ direction.

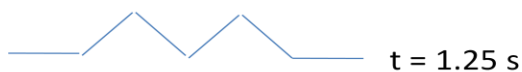
f) The tension in the rope is

$$F = \mu v^2 = \frac{m}{L} v^2 = 28.3 \text{ N.}$$

g) The average power transmitted by the wave is

$$P = \frac{1}{2} \sqrt{\mu F} \omega^2 A^2 = 0.39 \text{ W}$$

8. [50] Y&F Exercise 15.28 (merging of triangular pulses, section 15.6).



9. [40] Y&F Exercise 15.34 (standing waves, section 15.7).

a) The node to node distance is the same and the anti-node to anti-node distance, $\lambda/2 = 15.0 \text{ cm}$.

b) The traveling wave length is the same as the standing wave length, $\lambda = 30.0 \text{ cm}$.

The traveling wave amplitude is half the standing wave amplitude, $A_{travel} = A_{stand} / 2 = 0.425 \text{ cm}$.

The speed is $v = f\lambda = \frac{\lambda}{T} = 13.3 \text{ m/s}$.

c) The minimum transverse speed of a point at an anti-node is $v_{y,min} = 0$. The maximum speed is

$$v_{y,max} = \omega A_{stand} = 2\pi f A_{stand} = \frac{2\pi}{T} A_{stand} = 0.0712 \frac{m}{s}.$$

d) The shortest distance between a node and anti-node is $\lambda/4 = 7.50 \text{ cm}$.

10. [30] Y&F Exercise 15.39 (normal modes of a string, section 15.8).

a) The speed of transverse waves is $v = f\lambda = 96.0 \text{ m/s}$.

b) The tension in the wire is

$$F = \mu v^2 = \frac{m}{L} v^2 = 461 \text{ N}.$$

c) The maximum transverse speed is $v_{y,max} = \omega A_{stand} = 1.13 \frac{m}{s}$.

The maximum transverse acceleration is $a_{y,max} = \omega^2 A_{stand} = 426 \text{ m/s}^2$.