PH1140 D09 **Homework 3 Solution**

1. [30] Y&F 13.48.

a) Assuming a small angle displacement $\theta_{max} < 0.1\ rad$, the period is very nearly

$$T = 2\pi\sqrt{\frac{L}{g}} = 2.84\ s\ .$$

b) For the displacement $\theta_{max} = 30° = 0.524\ rad$ we use the first three terms of eq. 13.35

$$T = \left(2\pi\sqrt{\frac{L}{g}}\right)\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_{max}}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\theta_{max}}{2}\right) + \cdots\right] = 2.89\ s\ .$$

c) Part (b) is more accurate. The percent error of part (a) is

$$\frac{2.89 - 2.84}{2.89} = 0.02 = 2\%.$$

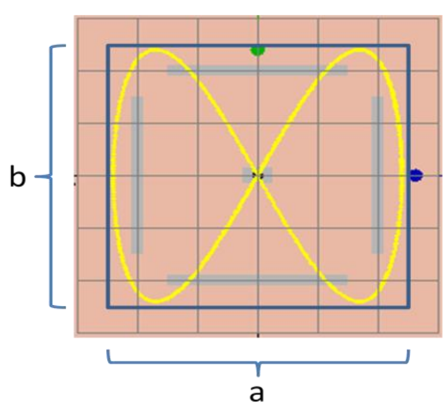
2. [20] You wish to construct a Lissajous pendulum that "visits" the four corners of a rectangle once a minute. The rectangle has length a and width b.

a) What amplitudes A_1 and A_2 are required?

Let a and b be the x and y dimensions of the rectangle, respectively. Let the center of the rectangle be the equilibrium position for both the x and y motions. Then the rectangle circumscribes the Lissajous figure, meaning that the outmost reaches of the Lissajous figure in both the x and y directions touch the rectangle. The amplitudes are then half the length and width of the rectangle, respectively. Letting A_1 be the amplitude of $x(t)$ and A_2 be the amplitude of $y(t)$, the amplitudes are

$$A_1 = \frac{a}{2}, \qquad A_2 = \frac{b}{2}.$$

b) What minimum frequencies f_1 and f_2 are required?



Inspection of French figure 2-14 reveals that the Lissajous figure defined by the ratio of frequencies 1: 2 and phase difference $\pi/2$ is the figure having the minimum frequencies that "visits" the four corners in the given period of time.

The Lissajous figure completes one cycle (is drawn once) during one beat period T_{beat}, which is related to the periods T_1 and T_2 of the x and y motions, respectively, by

$$T_{beat} = 60\ s = n_1 T_1 = n_2 T_2\ .$$

Since the ratio of frequencies is 1: 2, the ratio of the periods is 2: 1. In the figure above, the motion $x(t)$ has twice the period of motion $y(t)$, so the periods and minimum frequencies must be

$$T_1 = 60\ s\,, \qquad T_2 = 30\ s\,, \qquad f_1 = \frac{1}{0} Hz = 0.0167\ Hz\,, \qquad f_2 = \frac{1}{30} Hz = 0.0333\ Hz\,.$$

3. [60] You hang two pendulums of different lengths L_1 and L_2 in close proximity and from different heights so that the pendulum bobs "touch" once a minute. In one minute, the pendulums complete n_1 and n_2 cycles, respectively.

a) [30] Determine the lengths in terms of n_1 and n_2.
The pendulums have different lengths, therefore different frequencies, and will therefore touch once every beat period. The beat period is related to the pendulum periods by

$$T_{beat} = 60\ s = n_1 T_1 = n_2 T_2\,,$$

giving

$$T_1 = 2\pi\sqrt{\frac{L_1}{g}} = \frac{60}{n_1}\,, \qquad T_2 = 2\pi\sqrt{\frac{L_2}{g}} = \frac{60}{n_2}\,.$$

Solving for the lengths

$$L_1 = g\left(\frac{60}{2\pi\, n_1}\right)^2\,, \qquad L_2 = g\left(\frac{60}{2\pi\, n_2}\right)^2\,.$$

b) Give three idealizations that are required for "touches" to occur.

1) Neglect all forms of resistance (air, pivot).
2) When the pendulums touch they do so "infinitely" softly, so that they do not speed or slow each other's motions (otherwise, after the first touch their phases and periods would change and touches would become irregular at best, and non-existent at worst).
3) We are able to fabricate the lengths in an exact ratio of integers (otherwise the pendulums would never again precisely reach maximum displacements at the same moment and touch).

c) Give a set of initial conditions that ensure touches occur.
Start with the pendulums touching, that is at maximum displacements and at rest

$$\theta_1 = \theta_{1,max}\,, \qquad \theta_2 = -\theta_{2,max}\,, \qquad \dot{\theta}_1 = 0\,, \qquad \dot{\theta}_2 = 0\,.$$

d) Why must the ratio n_1/n_2 be a rational fraction?

Then, an integral (whole) number of periods of each pendulum occur during one beat period, and the pendulums touch again and again.

Coupled Oscillations

4. [50] Two identical pendulums A and B are connected by a spring of force constant $k = 1.017\ N/m$. Each pendulum has a length of $L = 0.4\ m$ and a mass of $m = 0.23\ kg$. Neglect the mass of the spring, and use gravitational acceleration $g = 9.8 m/s^2$.

a) What are the periods of the two normal oscillation modes of the coupled pendulums?

The normal mode frequencies of the coupled pendulums are

$$\omega_1 = \sqrt{\frac{g}{L}} = 4.95\ \frac{rad}{s}, \quad \omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}} = 5.77\ \frac{rad}{s}.$$

The periods are then

$$T_1 = \frac{2\pi}{\omega_1} = 1.269\ s, \quad T_2 = \frac{2\pi}{\omega_2} = 1.089\ s.$$

b) For the initial conditions $x_A = 0.02m,\ x_B = 0.02m,\ v_A = 0,\ v_B = 0$, determine the amplitudes and initial phases of the pendulum displacements $x_A(t)$ and $x_B(t)$. Hint: only one normal mode is involved.

Since $x_A = x_B$ and $v_A = v_B$ the motions of the two pendulums are identical, which is the case when both pendulums move in the first normal mode. We have then that

$$x_A = x_B = q_1 = A_1 \cos(\omega_1 t + \phi_1).$$

At maximum displacement a pendulum is stationary. This is precisely the initial state of pendulum A (and B), therefore the amplitude is simply

$$A_1 = x_A(t = 0) = 0.02\ m$$

and the phase is determined from

$$x_A(t = 0) = 0.02 = A_1 \cos(\omega_1 t + \phi_1) = 0.02 \cos(\phi_1)$$

$$\phi_1 = \cos^{-1}\left(\frac{0.02}{0.02}\right) = 0.$$

c) For the initial conditions $x_A = 0,\ x_B = 0,\ v_A = 0.173m/s,\ v_B = -0.173\ m/s$, determine the amplitudes and initial phases of the pendulum displacements $x_A(t)$ and $x_B(t)$. Hint: only one normal mode is involved.

The motions of the two pendulums are equal and opposite, which is the case when both pendulums move in the second normal mode. We have then that

$$x_A = -x_B = q_2 = A_2 \cos(\omega_2 t + \phi_2).$$

A pendulum moves at maximum speed when it passes through its equilibrium position. This is precisely the initial state of pendulum A (and B), therefore the amplitude is obtained simply from

$$v_A(t = 0) = 0.173\frac{m}{s} = \omega_2 A_2 = 5.77\ A_2$$

$$A_2 = 0.03\ m$$

using the normal frequency ω_2 from part (a). The phase is determined from

$$v_A(t = 0) = 0.173 = -\omega_2 A_2 \sin(\omega_2 t + \phi_2) = -0.173 \sin(\phi_2)$$

$$\phi_2 = \sin^{-1}\left(\frac{0.173}{-0.173}\right) = -\frac{\pi}{2}.$$

d) Starting with both pendulums in their equilibrium positions, one pendulum is given an initial displacement and then released. What is the time interval T_{beat} between successive maximum amplitudes of pendulum A? Hint: try using the coupled pendulum applet to study the motion, and consider the beat period as arising from the superposition of the two normal modes.

The initial position induces a motion in both pendulums that is a combination of both normal modes

$$x_A = \frac{q_1 + q_2}{2} = \frac{1}{2}[A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)]$$

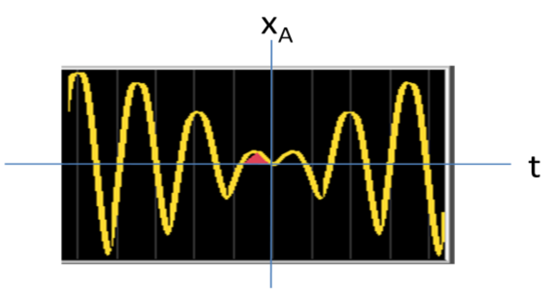
$$x_B = \frac{q_1 - q_2}{2} = \frac{1}{2}[A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)].$$

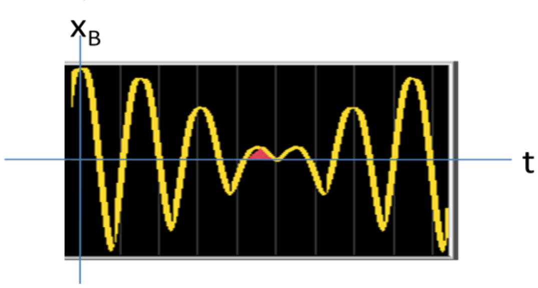
For both pendulums, the superposition produces beats of beat frequency and period

$$\omega_{beat} = \omega_1 - \omega_2 , \qquad T_{beat} = \frac{2\pi}{\omega_{beat}} = 7.62\ s.$$

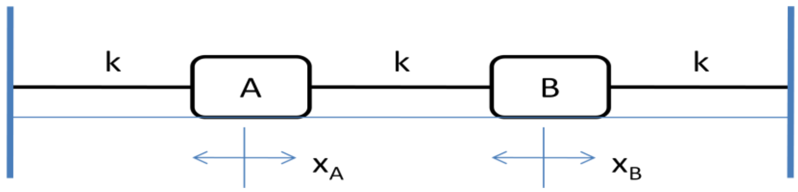
e) Sketch $x_A(t)$ and $x_B(t)$ over the time interval T_{beat} .

The images below were made with the phasor applet and show oscillations over one beat period T_{beat} for pendulums A and B. The B plot is offset to the right by $T_{beat}/2$, to indicate the relative phase of the A and B.





5. [50] Longitudinal Oscillations of Two Carts. Two carts A and B, both of mass m, are attached to three identical springs of force constant k, all mounted between two fixed posts. The carts are shown in their equilibrium positions in the figure below. The carts may oscillate longitudinally, that is, horizontally left and right. Let their displacements be x_A and x_B.



When displaced, the carts experience forces from the springs.

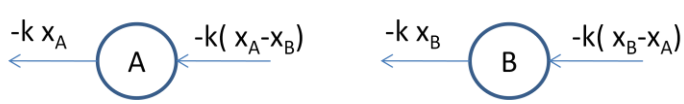
a) [30] Using free body diagrams and Newton's second law, sum the spring forces acting on each cart and show that

$$m\ddot{x}_A = -kx_A + k(x_B - x_A)$$

$$m\ddot{x}_B = -kx_B - k(x_B - x_A).$$

The carts are coupled oscillators, and the above equations of motion are coupled, that is, both involve x_A and x_B. There are two normal oscillation modes. In the first mode the carts have the same motion and satisfy the condition $x_A = x_B$. In the second mode the carts have opposite motions and satisfy the condition $x_A = -x_B$.

Solution: In the free body diagrams below, the arrows indicate the directions of the spring forces for positive displacements x_A and x_B, and positive $\Delta x = x_B - x_A$.



The force on A due to the leftmost spring is $-kx_A$, and the force on A due to the center spring is $-k(x_A - x_B)$. Note how the tension in the center spring (whether stretched or compressed) is a function of the change in its length $\Delta x = x_B - x_A$. The force on B due to the rightmost spring is $-kx_B$, and the force on B due to the center spring is $-k(x_B - x_A)$. Note how the center spring exerts equal but opposite forces on A and B, reflected in the signs of x_A and x_B. Inserting these forces into Newton's second law leads directly to the equations of motion above.

b) Substitute the first condition into the two equations of motion to decouple them, that is, form two equations, one involving only x_A and the other involving only x_B. From these, obtain the oscillation frequency ω_1 of the first mode.

Solution: Substituting the condition $x_A = x_B$ into the equation of motion for A to eliminate x_B gives

$$m\ddot{x}_A = -kx_A + k(x_A - x_A) = -kx_A, \quad \text{or} \quad \ddot{x}_A = -\frac{k}{m}x_A = -{\omega_1}^2 x_A.$$

Similarly, eliminating x_A from the equation of motion for B gives

$$m\ddot{x}_B = -kx_B + k(x_B - x_B) = -kx_B, \quad \text{or} \quad \ddot{x}_B = -\frac{k}{m}x_B = -\omega_1{}^2 x_B.$$

The decoupled equations have the same frequency $\omega_1{}^2 = k/m$.

c) Substitute the second condition into the two equations of motion to decouple them and obtain the oscillation frequency ω_2 of the second mode.

Solution: Substituting the condition $x_A = -x_B$ into the equation of motion for A to eliminate x_B gives

$$m\ddot{x}_A = -kx_A + k(-x_A - x_A) = -3kx_A, \quad \text{or} \quad \ddot{x}_A = -3\frac{k}{m}x_A = -\omega_2{}^2 x_A.$$

Similarly, eliminating x_A from the equation of motion for B gives

$$m\ddot{x}_B = -kx_B + k(-x_B - x_B) = -3kx_B, \quad \text{or} \quad \ddot{x}_B = -3\frac{k}{m}x_B = -\omega_2{}^2 x_B.$$

The decoupled equations have the same frequency $\omega_2{}^2 = 3k/m$.