PH1140 D09 **Study Guide Three** **2D and 3D Waves, Interactions, and Wave/Particle Duality**

Exam Preparation. You may bring one sheet of equations to the exam. Prepare this sheet well.

Key Concepts and Goals. In the third part of the course you study wave motion in two and three dimensions, wave-oscillator interactions, and wave/particle duality in nature. You study 2D waves on membranes and water surfaces, 3D sound waves in solids, liquids and gases, and 3D E&M waves.

Surface and Membrane Waves. Waves on a two dimensional surface such as an elastic membrane (for example, a drum head) satisfy the 2D wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

with wave speed $v = \omega/k = \sqrt{\mathcal{F}/\sigma}$, where $\mathcal{F}$ is the membrane tension per unit length of membrane cross section, and σ is the surface mass density (mass per unit area). Plane wave have the form

$$z(x, y, t) = A\cos\left(k_x x + k_y y - \omega t\right)$$

where the frequency and speed are related to two wave numbers by $k^2 = {k_x}^2 + {k_y}^2 = \omega^2/v^2$. The direction of the wave is given by the direction of the wave *vector* $\vec{\boldsymbol{k}} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, whose magnitude k is the wave number.

While the coordinate x locates a point on a string, the two coordinates $\{x, y\}$ locate a point on a surface. The vector $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ also locates a point on a surface. A point of constant phase

$$\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} - \omega t = k_x x + k_y y - \omega t = \varphi_{constant}$$

travels at wave speed $v = \omega/k$. A wave front is the set of points of constant phase, given by $\boldsymbol{k} \cdot \boldsymbol{r} - \omega t = \varphi_{constant}$. For a string, a wave front is a point moving along the string, but for a surface, the wave front is a line moving along the surface.

Due to diffraction (spreading out in all directions) plane waves landing on a barrier and passing through two slits each of width $s \ll \lambda$ and spaced a distance d apart, form interference zones at angles

$$\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right), \qquad n = 0,1,2,\ldots, \qquad constructive,\ \ crest + crest$$

$$\theta = \sin^{-1}\left(\frac{n\lambda}{2d}\right), \qquad n = 1,3,\ldots, \qquad destructive,\ \ crest + trough$$

These formulas are good for $r \gg d$ and $\gg \lambda$. For all r the exact description of interference is based on the path difference $r_{dif} = r_1 - r_2$ between the lengths of the wave paths r_1 and r_2 from slits 1 and 2. The phase difference between the two waves is $\varphi_{dif} = r_{dif}/\lambda$.

Circular waves are described by Bessel functions, which are beyond the scope of this course. However for large r and for $r \gg \lambda$, the circular waves are described approximately by

$$z(r, t) = \frac{A}{\sqrt{r}} e^{-i(kr \pm \omega t)} .$$

The average power P in units of W/m delivered by circular waves $z(r, t)$ on a membrane is proportional to the amplitude squared and falls off inversely with r

$$P_{avg} = \frac{\mathcal{F}\ k\ \omega\ A^2}{2r} = \frac{\sqrt{\sigma\mathcal{F}}\ \omega^2 A^2}{2r}\ , \qquad large\ \ r\ .$$

Water Waves. Plane (straight) surface waves are described by sinusoidal displacements of water particles given by

$\xi(x,t) = (Ak/\omega)cosh(ky)\cos(kx - \omega t)$ for the longitudinal displacement

$\eta(x,t) = (Ak/\omega)sinh(ky)\cos(kx - \omega t)$ for the transverse (vertical) displacement

for a wave traveling in the $+x$ direction, and where k is the wave number, ω is the frequency. These displacements amount to elliptical motion of the particles. For $y = h$, the displacements are the surface displacements (the visible ones)! The surface amplitude is given by

$$\eta_{max} = \frac{Ak}{\omega} sinh(kh).$$

The frequency is given by $\omega = \sqrt{gk\ tanh(kh)}$, where g is the gravitational acceleration and h is the depth of the water. The wave speed is $v = \omega/k = \sqrt{g/k\ \ tanh(kh)}$. Both frequency and wave speed are functions of the depth of the water. Clearly, dispersion exists, the longer wavelength (smaller wave number) waves traveling faster than the shorter wavelength waves. Wave speed increases with increasing depth, asymptotically approaching $v \simeq \sqrt{g/k}$ as $tanh(kh)$ approaches 1, which happens when $\lambda < h$. Also, for $\lambda \gg h$, the speed becomes $v \simeq \sqrt{gh}$.

The average energies of water waves are (from Elmore 6.5)

$$\bar{E} = \bar{K} + \bar{V} = \frac{1}{2}\rho g\ {\eta_{max}}^2.$$

Objectives. Be able to do the following:

1. Know the form of the 2D wave equation, and apply the solutions to problems involving waves. Identify each of the quantities that appear in these equations.

2. Work with the relations between frequency, period, wave number and wave speed.

4. For membrane waves, given some of the quantities: wave speed, tension, surface mass density, frequency and wave number, calculate some of the other quantities.

5. Understand and apply the dispersion relation for water waves.

6. Analyze and sketch the normal modes of membranes.

7. Know the difference and the relationship between traveling and standing waves on membranes.

8. Calculate the power of a string wave, and relate wave power to the power of the vibrator creating the wave, and the power lost to a damped oscillator under steady state conditions.

9. For 2D traveling waves, determine the direction of travel from the wave vector, and sketch wave fronts, showing correct orientation and spacing.

10. Calculate the frequency, wavelength and speed of water waves for various depths of water. Given wave length and water depth, determine whether or not tidal waves are present.