

Preparation. You may bring one sheet of equations to the exam. Prepare this sheet well.

Key Concepts and Goals. In the second part of the course you study superimposed and coupled oscillations of mechanical systems, and wave motion in continuous media of one dimension, that is longitudinal waves on a slinky and transverse waves a string.

Superimposed Oscillations. Two oscillations of the same frequency are described by phasors

$$z_1 = A_1 e^{i(\omega t + \phi_1)} \text{ and } z_2 = A_2 e^{i(\omega t + \phi_2)}.$$

These may be superimposed (added together) and are readily described by the sum of phasors

$$z = z_1 + z_2 = e^{i(\omega t + \phi_1)} [A_1 + A_2 e^{i(\phi_2 - \phi_1)}].$$

Two parallel superimposed oscillations are described by

$$x = x_1 + x_2 = A[\cos(\omega_1 t) + \cos(\omega_2 t)] = 2A \left[\cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \right]$$

with “beat” and oscillation frequencies

$$\omega_{beat} = \omega_1 - \omega_2 \text{ and } \omega_{osc} = \frac{\omega_1 + \omega_2}{2}.$$

If the ratio ω_1/ω_2 is a rational fraction, the superimposed oscillation will be periodic with beat period T an integer multiple of periods T_1 and T_2 :

$$T = n_1 T_1 = n_2 T_2, \text{ or } \tau = \frac{T}{2\pi} = \frac{n_1}{\omega_1} = \frac{n_2}{\omega_2}.$$

Two superimposed perpendicular oscillations form the vector displacement

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad x = A_1 \cos(\omega_1 t + \phi_1) \text{ and } y = A_2 \cos(\omega_2 t + \phi_2),$$

which generates Lissajous figures for ω_1/ω_2 a rational fraction.

Coupled Oscillations. The equations of motion of two pendulums coupled by a spring

$$\begin{aligned} \ddot{x}_A + (\omega_{pendulum}^2 + \omega_{spring}^2)x_A - \omega_{spring}^2 x_B &= 0 \\ \ddot{x}_B + (\omega_{pendulum}^2 + \omega_{spring}^2)x_B - \omega_{spring}^2 x_A &= 0 \end{aligned}$$

are easily solved using the normal mode coordinates

$$q_1 = x_A + x_B, \quad q_2 = x_A - x_B$$

to give the normal mode solutions

$$q_1 = C \cos \omega_1 t, \quad q_2 = D \cos \omega_2 t$$

with normal frequencies

$$\omega_1 = \omega_{pendulum} = \sqrt{g/l}, \quad \omega_2 = \sqrt{\omega_{pendulum}^2 + 2\omega_{spring}^2} = \sqrt{g/l + 2k/m}.$$

The longitudinal pendulum displacements are super-positions of the normal modes

$$x_A = (q_1 + q_2)/2, \quad x_B = (q_1 - q_2)/2.$$

[For \$N\$ coupled transverse oscillators](#), specifically N carts of mass m uniformly spaced a length l along an elastic string of length L and having a tension T , the n th normal mode of the p th cart is given by

$$y_n(p, t) = A_{pn} \cos(\omega_n t - \delta_n), \quad p = 1, 2, \dots, \quad n = 1, 2, \dots$$

with amplitudes and frequencies

$$A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right], \quad \omega_0^2 = T/ml.$$

By replacing cart number p with cart position $x = pl$ where l is the inter-cart distance, the normal mode solutions can be written as functions of position and time $\{x, t\}$

$$y_n(x, t) = A_n \sin(k_n x) \cos(\omega_n t - \delta_n), \quad k_n = n\pi/L, \quad L = l(N+1).$$

These solutions are *standing waves*, which can be shown to be combinations of two *traveling waves*

$$y_n(p, t) = \frac{1}{2} A_n [\sin(k_n x + \omega_n t - \delta_n) + \sin(k_n x - \omega_n t + \delta_n)]$$

traveling in opposite directions. Points of constant phase $k_n x \pm \omega_n t \mp \delta_n = \text{constant}$, travel along the string with wave speeds

$$v_n = \frac{\omega_n}{k_n}$$

This formula is the dispersion relation for waves on a string. We have crossed the threshold from oscillations to waves!

String Waves. Waves on strings are described by the

$$1D \text{ wave equation: } \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{d^2 y}{dt^2} = \frac{k^2}{\omega^2} \frac{\partial^2 y}{\partial t^2} = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}, \quad v = \frac{\omega}{k} = \sqrt{F/\mu}$$

$$1D \text{ wave function: } y(x, t) = 2A \sin(kx) \cos(\omega t) = A[\sin(kx + \omega t) + \sin(kx - \omega t)]$$

$$1D \text{ wave function (phasor): } z(x, t) = A e^{ikx} e^{\pm i\omega t} = A e^{i(kx \pm \omega t)}$$

and the period T , frequency f , wavelength λ , wave number k , and wave speed v :

$$T = 1/f = 2\pi/\omega, \quad 2\pi/\lambda = k, \quad v = \lambda/T = \lambda f = \omega/k$$

The power delivered by the string force of one part of the string acting on the other part, the two parts meeting at a point x , is, from $P = \mathbf{F} \cdot \mathbf{v}$ and with $y(x, t) = A \cos(kx - \omega t)$

$$P(x, t) = F_y(x, t) v_y(x, t) = -F \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} = Fk\omega A^2 \sin^2(kx - \omega t).$$

Using $\omega = vk$ and $v^2 = F/\mu$ gives the alternate form $P(x, t) = \sqrt{\mu F} \omega^2 A^2 \sin^2(kx - \omega t)$.

The maximum and average powers are

$$P_{max} = \sqrt{\mu F} \omega^2 A^2, \quad P_{avg} = \sqrt{\mu F} \omega^2 A^2 / 2.$$

Normal Modes. Normal modes of a string are standing waves having an integral multiple of half-waves

$$L = n\pi/k = n\lambda/2, \quad n = 1, 2, \dots$$

The n th mode has a wavelength and frequency given by

$$\lambda_n = 2L/n, \quad f_n = v/\lambda_n = nv/2L = 1/2L \sqrt{F/\mu}.$$

The n th mode wave function has the form

$$y_n(x, t) = A_n \sin(k_n x) \sin(\omega_n t).$$

For longitudinal waves on a slinky, each element of the slinky experiences a longitudinal displacement ξ about an equilibrium position x , having the form (for a traveling wave)

$$\xi(x, t) = A \cos(kx - \omega t).$$

As with transverse waves, the speed of the longitudinal wave is given by

$$v = \sqrt{F/\mu}.$$

Specific Objectives.

For superimposed and coupled oscillations:

1. Distinguish between superimposed and coupled oscillations.
2. Analyze superimposed oscillations, both parallel and perpendicular, with same and different frequencies, and describe them using phasors, both algebraically and graphically.
3. Analyze parallel superimposed oscillations in terms of interference and beats.
4. Use initial conditions to determine the solutions of coupled oscillators.
5. Analyze coupled oscillations in terms of their normal modes of oscillation.
6. Analyze large- N coupled systems in terms of collective motion tending toward standing and traveling wave motion, characterized by wavelength, wave speed and dispersion.

For 1D waves:

7. Know the form of the wave equation, and apply its solution to problems involving waves. Identify each of the quantities that appear in these equations.
8. Work with the relations between frequency, period, wave number and wave speed.
9. Given several of the quantities wave speed, tension, linear mass density, frequency and wave number, calculate the others.
10. Analyze and sketch the normal modes of a slinky and a string.
11. Know the difference and the relationship between traveling and standing waves.
12. Calculate the power of a string wave, and relate wave power to the power of the vibrator creating the wave, and the power lost to a damped oscillator under steady state conditions.