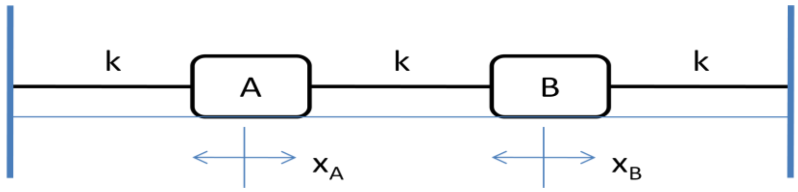
PH1140 B08 Written Homework Four – Coupled Oscillators Due November 21, 2008

1. Longitudinal Oscillations of Two Carts. Two carts A and B, both of mass m, are attached to three identical springs of force constant k, all mounted between two fixed posts. The carts are shown in their equilibrium positions in the figure below. The carts may oscillate longitudinally, that is, horizontally left and right. Let their displacements be x_A and x_B.



When displaced, the carts experience forces from the springs.

a) Using free body diagrams and Newton's second law, sum the spring forces acting on each cart and show that

$$m\ddot{x}_A = -kx_A + k(x_B - x_A)$$

$$m\ddot{x}_B = -kx_B - k(x_B - x_A).$$

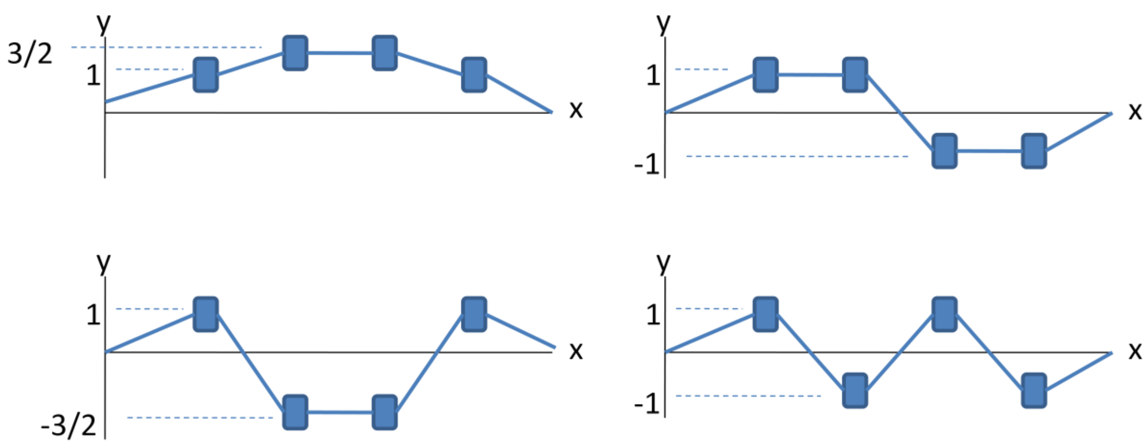
The carts are coupled oscillators, and the above equations of motion are coupled, that is, both involve x_A and x_B. There are two normal oscillation modes. In the first mode the carts have the same motion and satisfy the condition $x_A = x_B$. In the second mode the carts have opposite motions and satisfy the condition $x_A = -x_B$.

b) Substitute the first condition into the two equations of motion to decouple them, that is, form two equations, one involving only x_A and the other involving only x_B. From these, obtain the oscillation frequency ω_1 of the first mode.

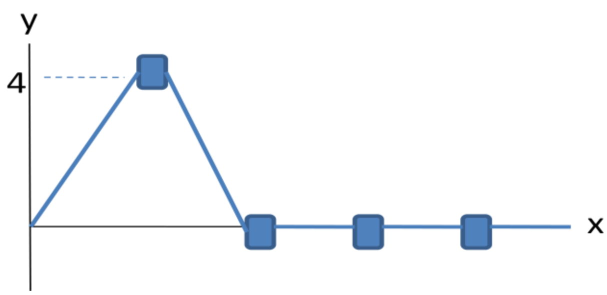
c) Substitute the second condition into the two equations of motion to decouple them and obtain the oscillation frequency ω_2 of the second mode.

2. Transverse Oscillations of Four Carts. From your experience in lab, you know that four carts on elastic string have four transverse normal oscillation modes. Suppose that the four modes can be created, respectively, by the four sets of initial displacements shown below. For example, the first mode displacements are

$$y_1 = 1, \qquad y_2 = \frac{3}{2}, \qquad y_3 = \frac{3}{2}, \qquad y_4 = 1.$$



a) Add the displacements to show that the combination of the four sets of displacements is equivalent to the set of displacements shown below.



From part (a), it is clear that the initial displacement of a single cart is equivalent to a combination of initial displacements, and "contains" a little bit of each of the four normal modes. In lab, you observed how, when a single cart is given an initial displacement and released, its motion is carried to the second cart, then the third, and finally the fourth, forming wave motion. This wave motion is a superposition of the four normal modes, and is therefore a superposition of traveling waves involving four different wave speeds.

b) Starting with the four carts in their equilibrium positions, you displace the first cart and release it to initiate transverse motion. Assume uniform cart spacing l such that the string length is given by $L = (N + 1)l$, where $N = 4$. In terms of the string tension T, the cart spacing l, and the cart mass m, how long does it take for the fourth cart to "feel" the motion?