

Preparation. You may bring one sheet of equations to the exam. Prepare this sheet well.

Key Concepts and Goals. In the third part of the course you study wave motion in mechanical systems of one and two dimensions; longitudinal and transverse waves on strings, and transverse waves on membranes and water surfaces.

String Waves. Waves on strings are described by the

$$1D \text{ wave equation: } \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{d^2 y}{dt^2} = \frac{k^2}{\omega^2} \frac{\partial^2 y}{\partial t^2} = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}, \quad v = \frac{\omega}{k} = \sqrt{(F/\mu)}$$

$$1D \text{ wave function: } y(x, t) = 2A \sin(kx) \cos(\omega t) = A[\sin(kx + \omega t) + \sin(kx - \omega t)]$$

$$1D \text{ wave function (phasor): } z(x, t) = Ae^{ikx} e^{\pm i\omega t} = Ae^{i(kx \pm \omega t)}$$

and the period T , frequency f , wavelength λ , wave number k , and wave speed v :

$$T = 1/f = 2\pi/\omega, \quad 2\pi/\lambda = k, \quad v = \lambda/T = \lambda f = \omega/k$$

The power delivered by the string force of one part of the string acting on the other part, the two parts meeting at a point x , is, from $P = \mathbf{F} \cdot \mathbf{v}$

$$P(x, t) = F_y(x, t)v_y(x, t) = -F \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} = Fk\omega A^2 \sin^2(kx - \omega t).$$

Using $\omega = vk$ and $v^2 = F/\mu$ gives the alternate form $P(x, t) = \sqrt{\mu F} \omega^2 A^2 \sin^2(kx - \omega t)$.

The maximum and average powers are

$$P_{max} = \sqrt{\mu F} \omega^2 A^2, \quad P_{avg} = \sqrt{\mu F} \omega^2 A^2 / 2.$$

Normal Modes. Normal modes of a string are standing waves having an integral multiple of half-waves

$$L = n\pi/k = n\lambda/2, \quad n = 1, 2, \dots$$

The n th mode has a wavelength and frequency given by

$$\lambda_n = 2L/n, \quad f_n = v/\lambda_n = nv/2L = 1/2L \sqrt{(F/\mu)}.$$

The n th mode wave function has the form

$$y_n(x, t) = A_n \sin(k_n x) \sin(\omega_n t).$$

For longitudinal waves on a slinky, each element of the slinky experiences a longitudinal displacement ξ about an equilibrium position x , having the form (for a traveling wave)

$$\xi(x, t) = A \cos(kx - \omega t).$$

As with transverse waves, the speed of the longitudinal wave is given by

$$v = \sqrt{(F/\mu)}.$$

Membrane Waves. Waves on a two dimensional surface such as an elastic membrane (for example, a drum head) satisfy the 2D wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

with wave speed $v = \sqrt{f/\sigma}$, where f is the membrane tension per unit length of membrane cross section, and σ is the surface mass density (mass per unit area). The wave solutions have the form

$$z(x, y, t) = A \cos(k_x x + k_y y - \omega t)$$

where the frequency and speed are related to two wave numbers by $k^2 = k_x^2 + k_y^2 = \omega^2/v^2$.

The direction of the wave is given by the direction of the *vector wave* $\vec{k} = k_x \hat{x} + k_y \hat{y}$, whose magnitude k is the wave number.

While the coordinate x locates a point on a string, the two coordinates $\{x, y\}$ locate a point on a surface. The vector $\vec{r} = x\hat{x} + y\hat{y}$ also locates a point on a surface. A point of constant phase

$$\vec{k} \cdot \vec{r} - \omega t = k_x x + k_y y - \omega t = \varphi_{constant}$$

travels at wave speed. A wave front is the set of points of constant phase at a fixed time, given by $\vec{k} \cdot \vec{r} - \omega t_{fixed} = \varphi_{constant}$. For a string, a wave front is a point moving along the string, but for a surface, the wave front is a line moving along the surface.

Water Waves. Plane (straight) surface waves are described by sinusoidal displacements of water particles given by

$$\begin{aligned} \xi(x, t) &= (Ak/\omega) \cosh(ky) \cos(kx - \omega t) \text{ for the longitudinal displacement} \\ \eta(x, t) &= (Ak/\omega) \sinh(ky) \cos(kx - \omega t) \text{ for the transverse (vertical) displacement} \end{aligned}$$

for a wave traveling in the $+x$ direction, and where k is the wave number, ω is the frequency. These displacements amount to elliptical motion of the particles. For $y = h$, the displacements are the surface displacements (the visible ones)! The surface amplitude is given by

$$\eta_{max} = A \sqrt{\frac{k}{2g} \sinh(2kh)}.$$

The frequency is given by $\omega = \sqrt{gk \tanh(kh)}$, where g is the gravitational acceleration and h is the depth of the water. The wave speed is $v = \omega/k = \sqrt{g/k \tanh(kh)}$. Both frequency and wave speed are functions of the depth of the water. Clearly, dispersion exists, the longer wavelength (smaller wave number) waves traveling faster than the shorter wavelength waves. Wave speed increases with increasing depth, asymptotically approaching $v \approx \sqrt{g/k}$ as $\tanh(kh)$ approaches 1, which happens when $\lambda < h$. Also, for $\lambda \gg h$, the speed becomes $v \approx \sqrt{gh}$.

The average energies of water waves are (taken from Elmore 6.5)

$$\bar{E} = \bar{K} + \bar{V} = \frac{1}{2} \rho g \left(\frac{Ak}{\omega} \sinh(kh) \right)^2.$$

Objectives. Be able to do the following:

1. Know the form of the 1D and 2D wave equations, and apply their respective solutions to problems involving waves. Identify each of the quantities that appear in these equations.
2. Work with the relations between frequency, period, wave number and wave speed, for both string and membrane waves.
3. For string waves, given some of the quantities: wave speed, tension, linear mass density, frequency and wave number, calculate some of the other quantities.
4. For membrane waves, given some of the quantities: wave speed, tension, surface mass density, frequency and wave number, calculate some of the other quantities.
5. Understand and apply the dispersion relation for water waves.
6. Analyze and sketch the normal modes of strings and membranes.
7. Know the difference and the relationship between traveling and standing waves on strings and membranes.
8. Calculate the power of a string wave, and relate wave power to the power of the vibrator creating the wave, and the power lost to a damped oscillator under steady state conditions.
9. For 2D traveling waves, determine the direction of travel from the wave vector, and sketch wave fronts, showing correct orientation and spacing.
10. Calculate the frequency, wavelength and speed of water waves for various depths of water. Given wave length and water depth, determine whether or not tidal waves are present.