

**Preparation.** You may bring one sheet of equations to the exam. Prepare this sheet well.

**Key Concepts and Goals.** In the second part of the course you study forced, superimposed and coupled oscillations of mechanical systems.

**Forced Oscillations.** A mass-spring system subjected to a time-varying force  $F(t)$  obeys to equation

$$ma = -kx - bv + F(t),$$

which leads to the equation of motion

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)/m.$$

We often work with the sinusoid forcing function

$$F(t) = F_0 \cos \omega t = \text{re}(F_0 e^{i\omega t}).$$

Then the forced solution takes the form (with  $\omega_0$  the free or natural frequency)

$$x = A \cos(\omega t - \delta) = \text{re}(A e^{i(\omega t - \delta)})$$

$$A = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2]^{1/2}}, \quad \tan \delta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

The quality factor  $Q = \omega_0/2\gamma$  quantifies the ability of the system to resonate with the applied force.

**Superimposed Oscillations.** Two oscillations of the same frequency are described by phasors

$$z_1 = A_1 e^{i(\omega t + \varphi_1)} \quad \text{and} \quad z_2 = A_2 e^{i(\omega t + \varphi_2)}.$$

These may be superimposed (added together) and are readily described by the sum of phasors

$$z = z_1 + z_2 = e^{i(\omega t + \varphi_1)} [A_1 + A_2 e^{i(\varphi_2 - \varphi_1)}].$$

Two parallel oscillations of different frequencies may be superimposed and described by

$$x = x_1 + x_2 = A[\cos(\omega_1 t) + \cos(\omega_2 t)] = 2A \left[ \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \right]$$

with the “beat” and “combo” frequencies

$$\omega_{beat} = \frac{\omega_1 - \omega_2}{2} \quad \text{and} \quad \omega_{combo} = \frac{\omega_1 + \omega_2}{2}.$$

If the two frequencies are both rational fractions, the combined oscillation will be periodic, with a period  $T$  that is an integer multiple of both periods  $T_1$  and  $T_2$ :

$$T = n_1 T_1 = n_2 T_2, \quad \text{or} \quad \tau = \frac{T}{2\pi} = \frac{n_1}{\omega_1} = \frac{n_2}{\omega_2}.$$

Two perpendicular oscillations may be superimposed, giving rise to the vector displacement

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad x = A_1 \cos(\omega_1 t + \varphi_1) \quad \text{and} \quad y = A_2 \cos(\omega_2 t + \varphi_2).$$

We have already seen that circular motion is of this form.

**Coupled Oscillations.** The equations of motion of two pendulums coupled by a spring

$$\begin{aligned} \ddot{x}_A + (\omega_{pendulum}^2 + \omega_{spring}^2)x_A - \omega_{spring}^2 x_B &= 0 \\ \ddot{x}_B + (\omega_{pendulum}^2 + \omega_{spring}^2)x_B - \omega_{spring}^2 x_A &= 0 \end{aligned}$$

are easily solved using the normal mode coordinates

$$q_1 = x_A + x_B, \quad q_2 = x_A - x_B$$

to give the normal mode solutions

$$q_1 = C \cos \omega_1 t, \quad q_2 = D \cos \omega_2 t$$

with normal frequencies

$$\omega_1 = \omega_{\text{pendulum}} = \sqrt{g/l}, \quad \omega_2 = \sqrt{\omega_{\text{pendulum}}^2 + 2\omega_{\text{spring}}^2} = \sqrt{g/l + 2k/m}.$$

The original longitudinal pendulum displacements are super-positions of the normal modes

$$x_A = (q_1 + q_2)/2, \quad x_B = (q_1 - q_2)/2.$$

[For  \$N\$  coupled transverse oscillators](#), specifically  $N$  carts of mass  $m$  uniformly spaced a length  $l$  along an elastic string of length  $L$  and having a tension  $T$ ), the  $n$ th normal mode of the  $p$ th cart is given by

$$y_n(p, t) = A_{pn} \cos(\omega_n t - \delta_n), \quad p = 1, 2, \dots, \quad n = 1, 2, \dots$$

with amplitudes and frequencies

$$A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right], \quad \omega_0^2 = T/ml.$$

By replacing cart number  $p$  with cart position  $x = pl$  where  $l$  is the inter-cart distance, the normal mode solutions can be written as functions of position and time  $\{x, t\}$

$$y_n(x, t) = A_n \sin(k_n x) \cos(\omega_n t - \delta_n), \quad k_n = n\pi/L, \quad L = l(N+1).$$

These solutions are *standing waves*, which can be shown to be combinations of two *traveling waves*

$$y_n(p, t) = \frac{1}{2} A_n [\sin(k_n x + \omega_n t - \delta_n) + \sin(k_n x - \omega_n t + \delta_n)]$$

traveling in opposite directions. Points of constant phase  $k_n x \pm \omega_n t \mp \delta_n = \text{constant}$ , travel along the string with wave speeds

$$v_n = \frac{\omega_n}{k_n}$$

This formula is the dispersion relation for waves on a string. We have crossed the threshold from oscillations to waves!

**Specific Objectives.** Be able to do the following:

1. Analyze forced oscillations and characterize system behavior in terms of phase and resonance, quantified by  $A$ ,  $\delta$  and  $Q$ .
2. Distinguish between superimposed and coupled oscillations.
3. Analyze superimposed oscillations, both parallel and perpendicular, with same and different frequencies, and describe them using phasors, both algebraically and graphically.
4. Analyze parallel superimposed oscillations in terms of interference and beats.
5. Use initial conditions to determine the solutions of coupled oscillators.
6. Analyze coupled oscillations in terms of their normal modes of oscillation.
7. Analyze large- $N$  coupled systems in terms of collective motion tending toward standing and traveling wave motion, characterized by wavelength, wave speed and dispersion.