Like charged Styrofoam cups, bar magnets exert forces on one another, as if they possessed invisible hands that reach across empty space.

They also exert torques on one another, as can be seen by taking two bar magnets, hanging one from a string, and holding the other in your hand and moving it around.

Recall from mechanics that when two bodies bump into each other, they exert equal and opposite contact forces on one another

$$\vec{F}_{AonB} = -\vec{F}_{BonA}$$

If the force is directed through the center of mass of the body then there is no torque. If the force is "oblique" it exerts a torque $\tau = rF$

$$\tau = \vec{r} \times \vec{F} = rF\sin\theta\,\hat{u}_{\perp} = rF_{\perp}\hat{u}_{\perp}$$

where r is the "lever", the distance between the center of mass and the point of application of the force, and $F_{\perp}$ is the component of the force perpendicular to the lever.

We can demonstrate that two bar magnets exert both "body centered" forces as well as torques.

We say a charged cup has electric charge q, and this charge creates an invisible electric field in space all around the cup. The electric field exerts forces on other charged cups.

Similarly, we say that a bar magnet has a kind of "magnetic charge", called the magnetic dipole moment μ. We way the magnetic dipole creates an invisible magnetic field.

Iron filings sprinkled around a bar magnet (see wiki "magnetic field") reveal the field.

At a distance, a bar magnet looks like a "point" magnetic dipole, and its field lines look like the field of an electric dipole (sketch).

The magnetic field exerts forces and torques on other magnets, and as you will see, it also exerts forces on moving charges!

Like the bar magnet, the Earth has a magnetic dipole and a magnetic field, and this field exerts a torque on the needle of compass; the needle is a tiny bar magnet.

An electron possesses a magnetic dipole and a magnetic field (sketch).

Early experimenters found that a loop of wire carrying an electric current has a magnetic dipole and a magnetic field (sketch).

It makes you wonder: are there tiny loops of current inside of other magnets such as the earth, bar magnet and electron? The answer is yes, yes and no. The source of the electron's magnetic dipole moment, known as the Bohr magneton, is a mystery.

The dipole fields of these four magnets, the bar, Earth, electron and electromagnet are similar in shape; the same sketch applies to all.

The dipole has an axis, and its field has cylindrical symmetry about the axis.

At the start of the course, we talked about a point charge $\boldsymbol{q}$, then defined its electric field in terms of q

$$E \propto \frac{q}{r^2}$$

Then we talked about the force of the electric field on other charges.

Similarly we can talk about a "point" magnet's magnetic "charge" μ and its field B

$$B \propto \frac{\mu}{r^3}$$

We worked with a uniform electric field. Similarly, if you choose a small region near one face of a magnet, the B field is uniform. You can also choose a huge bar magnet whose faces are the size of this lecture hall.

A uniform field magnetic field may be described by

$$\vec{B} = B\hat{k}\,, \qquad units\ of\ Tesla\ [T]$$

A uniform magnetic field exerts a force on a moving point charge $\boldsymbol{q}$, moving with velocity $\vec{\boldsymbol{v}}$

$$\vec{F} = q\vec{v} \times \vec{B} = qvB\sin\theta\,\hat{u}_{\perp}$$

where θ is the angle between $\vec{v}$ and $\vec{B}$. The force acts in a direction $\hat{u}_{\perp}$ that is perpendicular to both the force and velocity. If we put the velocity and magnetic field vectors in the plane of the page or chalkboard, the force either points directly out of the page or into the page.

From this formula we see that the unit of Tesla is related to other units by

$$[B] = \left[\frac{F}{qv}\right] = \frac{N}{C \cdot m/s}$$

Compare the magnetic force to the electric force

$$\vec{F} = q\vec{E}$$

The electric force and field lie along the same line. Not so for the magnetic force and field.

In mechanics you used the right hand rule to determine whether a torque was clockwise (into the page) or counter-clockwise (out of the page). Here we use the right hand rule to determine if the magnetic force is "out of" or "into" the page. That is, does your right thumb point out of or into the page?

In terms of the vector components the formula expands into nine cross products of unit vectors

$$\vec{F} = q\vec{v} \times \vec{B} = q\left(v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\right) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

From the definition of the cross product $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{u}_\perp$ we quickly find

$$\vec{\imath} \times \vec{\imath} = (1)(1)\sin(0)\,\hat{u}_\perp = 0\,, \qquad \vec{\imath} \times \vec{\jmath} = (1)(1)\sin(90°)\,\hat{u}_\perp = \hat{k}\,, \qquad etc$$

You can use one of the popular mnemonic devices, but a little practice makes you handy with these.

For example if $\vec{v} = v\hat{\imath}$ and $\vec{B} = -B\hat{\jmath}$ then $\vec{F} = q\vec{v} \times \vec{B} = -qvB(\vec{\imath} \times \vec{\jmath}) = -qvB\hat{k}$. The force points into the page (chalkboard).

The component of the velocity that is perpendicular to the field is the only part that produces force

$$F = qB(v\sin\theta) = qBv_\perp$$

Knowing this can save you work; if you are working an exercise and see that the velocity and magnetic field vectors are parallel, you know the force is zero (there is no perpendicular component of $\vec{v}$).

A uniform magnetic field causes a moving point charge to move in a circle.

This is so because the force created by the field is perpendicular to the velocity.

A centripetal force has the same property

$$\vec{F} = m\vec{a} = -\frac{mv^2}{r}\hat{r}\,, \qquad \vec{F} \perp \vec{v}$$

Let $\vec{v}$ and $\vec{F}$ both lie in the plane of the page, let $\vec{B} = B\hat{k}$, and let $\vec{B} \perp \vec{v}$. Then $\sin\theta = 1$ and

$$\vec{F} = q\vec{v} \times \vec{B} = qvB\sin\theta\,\hat{u}_\perp = -qvB\hat{r}$$

Equating the two forces and solving for r

$$F = qvB = \frac{mv^2}{r}, \qquad \rightarrow \; r = \frac{mv}{qB}$$

we find the radius of the circle. The stronger the field is, the smaller the circle.

In 1897 Thompson used the above formula to infer that a cathode ray is a stream of electrons, and he measured the ratio of charge to mass of the electron. From the preceding formula (for a uniform field)

$$\frac{e}{m} = \frac{v}{rB}, \qquad (1)$$

By accelerating the electrons through a known potential V, the electron speed could be determined using the work-energy theorem $\Delta U = \Delta K$

$$eV = \frac{mv^2}{2}\,, \qquad \rightarrow \quad v^2 = \frac{e}{m}2V, \qquad (2)$$

Squaring both sides of equation (1) and substituting for v^2 from equation (2)

$$\left(\frac{e}{m}\right)^2 = \left(\frac{1}{rB}\right)^2 v^2 = \left(\frac{1}{rB}\right)^2 \frac{e}{m}2V\,, \qquad \rightarrow \quad \frac{e}{m} = \left(\frac{1}{rB}\right)^2 2V$$

gives a formula for e/m in terms of r, B and V.

Thompson measured the deflection of the electron beam, in effect he measured r, and computed e/m.

You might guess that a magnetic field exerts a force on a current carrying wire, since a magnetic field exerts a force on a moving charge, and since a current in a wire is a stream of moving charges.

Consider a uniform magnetic field $\vec{B}$ pointing into the page, a straight wire laying in the plane of the page, having cross section A and carrying current I consisting of particles of charge q moving with drift speed v_d along the wire. The force acting on a charge is perpendicular to both the wire and the field

$$F = qv_dB$$

The total free charge Q in a segment of wire of length L is

$$Q = qN = qnAL$$

where n is the number density of free charges.

The total force acting on the segment of wire (for angle $\theta = 90°$ between velocity and field)

$$F = Qv_dB = (qnAL)v_dB = ILB\,, \qquad \leftarrow \quad I = JA = nqv_dA$$

The more general formula takes into account the angle between wire and field

$$\vec{F} = I\vec{L} \times \vec{B}\,, \qquad d\vec{F} = I\,d\vec{L} \times \vec{B}\,, \qquad \vec{F} = \int d\vec{F}$$

For curvilinear wires the differential form $d\vec{F}$ may be used to sum the force over a length of wire. It may also be used to approximate the field created by a short segment $\Delta\vec{L}$ of a curved wire

$$\Delta\vec{F} \cong I\,\Delta\vec{L} \times \vec{B}$$