The Capacitor's Potential Energy. A capacitor C charged to a voltage V has charge

$$Q = CV\,, \qquad V = \frac{Q}{C}$$

Let's examine the charging process. At an intermediate stage of charging let the voltage be v

$$v = \frac{q}{C}$$

During the charging process both v and q increase. Both start at zero. When the capacitor voltage v reaches the applied voltage $v = V$, the charge reaches $q = Q$.

As an analogy, picture a cup being filled with water. As you lift a bit of water up and place in on top, the level rises. The next bit of water will have to be lifted a little higher, and so on.

Same with the capacitor; as you add a little bit of charge, the voltage rises. The next bit of charge will have to be "raised" through a higher voltage.

At the intermediate stage, it takes effort (work) to "lift" an additional infinitesimal element of charge dq from the negative plate to the positive plate, because the charge is being lifted through the potential v. The work dW required to lift dq is

$$dW = v\,dq = \frac{q}{C}dq$$

The total work required to charge the capacitor from $q = 0$ to $q = Q$ is the infinite sum

$$W = \int_0^W dW = \frac{1}{C}\int_0^Q q\ dq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q = \frac{Q^2}{2C} = U$$

This is the potential energy stored in the capacitor.

This is similar to a spring; the work required to stretch a spring equals the potential energy of the spring.

Substituting $Q = CV$ gives two other forms

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

One formula may be more convenient that another to calculate a capacitor's energy; it depends on which of the quantities Q, C, V are known.

Using Capacitor Energy. How much energy is stored in a capacitor?

$$C = 1F, \qquad V = 3.0\,V, \qquad U = 0.5(1F)(3V)^2\,J = 4.5\,J$$

Camera flash units charge a capacitor and then discharge it through a light bulb. When the unit flashes, the energy stored in the capacitor is released in the form of radiant energy (a flash of light). Flash units waste some of the energy in the form of heat, so U is an upper limit for estimating the radiant energy.

If a capacitor stores $U = 4.5\ J$ of energy, and the energy is released in the bulb in $t = 1\ ms$, the power is

$$P = \frac{U}{t} = 4500\ W$$

Energy density of the Electric Field. Remember how we began talking about the force between two charged Styrofoam cups? Then we came up with the wacky idea that a charge creates an invisible electric field in space, and that invisible field exerts a force on another charge. Did we stop there? No, we kept repeating this notion in the hopes that you would start to believe the electric field is real.

Now, just when you are starting to get used to the idea of an invisible electric field, we spring another wacky idea on you, that energy is stored in the field.

A parallel plate capacitor has potential energy U

$$U = \frac{1}{2}CV^2, \qquad C = \frac{\epsilon_0 A}{d}, \qquad V = Ed$$

If we substitute from our previously derived expressions for capacitance C and electric field strength E, and divide by the volume $= Ad$ of space inside the capacitor (the space where the uniform electric field lives), we obtain u, the "energy density of the electric field"

$$u = \frac{U}{Ad} = \frac{\epsilon_0}{2}E^2, \qquad units\ of\ \frac{J}{m^3}$$

We can store energy in the electric field, in empty space! Notice how the energy density is proportional to the square of the field strength. Double the field strength and quadruple the energy density.

Example: You pet and charge your cat to $Q = 9\ nC$. The field strength and energy density at a distance of one cat whisker is

$$r = 0.07m, \qquad E = \frac{kQ}{r^2} = 16.5\frac{kV}{m}, \qquad u = 0.00121\frac{J}{m^3}$$

What can you do with energy density of the field? For one thing, if a field changes, for example strengthens or weakens, we immediately know that energy is being gained or lost by the field. Since energy is conserved (neither created nor destroyed), we know that the energy is being somehow transferred in or out of the field; energy is exchanged between the field and some other physical thing such as the battery that charges a capacitor, or the flash bulb into which energy is dumped.

When the battery charges the capacitor, we say the battery pumps energy into the field (the electric field inside the capacitor).

When the capacitor discharges through a light bulb, we say that the field (of the capacitor) dumps its energy into the bulb.

The notion of energy density of the electric field plays a big role in the study of electromagnetic fields and waves, a branch of study that is beyond the scope of this course. If you pursue your E&M studies at the sophomore level and beyond, you will encounter the energy density often.

Ratio of the voltages of two series capacitors. Series capacitors have the same charge Q

$$Q = C_1 V_1 \,, \qquad Q = C_2 V_2$$

Since the Q's are equal we obtain

$$C_1 V_1 = C_2 V_2 \,, \qquad \Rightarrow \quad \frac{V_2}{V_1} = \frac{C_1}{C_2}$$

The voltages are in inverse proportion to the capacitances.

Example: you apply $V = 12\ V$ to series capacitors C_1 and $C_2 = 5C_1$. What are the capacitor voltages? First, we know the sum of the voltages has to equal V. So let's apply our new handy nifty formula

$$\frac{V_2}{V_1} = \frac{C_1}{C_2} = \frac{C_1}{5C_1} = \frac{1}{5}, \qquad V_1 = 5V_2, \qquad 12V = V_1 + V_2 = 5V_2 + V_2 = 6V_2, \qquad V_2 = \frac{12V}{6} = 2V$$

Ratio of the charges of two parallel capacitors. Parallel capacitors have the same voltage

$$V = \frac{Q_1}{C_1} \,, \qquad V = \frac{Q_2}{C_2}$$

Since the two V's are equal we obtain

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \,, \qquad \Rightarrow \quad \frac{Q_1}{Q_2} = \frac{C_1}{C_2}$$

The charges are in direct proportion to the capacitances.

You can form the ratio of the energies of two capacitors having the same voltage, or of two capacitors having the same charge

$$U_1 = \frac{1}{2} C_1 V^2, \qquad U_2 = \frac{1}{2} C_2 V^2, \qquad \frac{U_1}{U_2} = \frac{C_1}{C_2} \,, \qquad \textit{same } V$$

$$U_1 = \frac{1}{2} \frac{Q^2}{C_1}, \qquad U_2 = \frac{1}{2} \frac{Q^2}{C_2}, \qquad \frac{U_1}{U_2} = \frac{C_2}{C_1} \,, \qquad \textit{same } Q$$

Why do I bother mentioning all of these useful formulas? Who knows, they might come in handy on your homework, possibly even your upcoming exam, or at home when you wish to decide which of your favorite capacitors to use to store energy.