The electric potential of a point charge

The work W done by a field is the negative of the change in potential energy

$$\Delta U = -W_{field} = -\vec{F}_{field} \cdot \Delta\vec{r} = -q\vec{E} \cdot \Delta\vec{r}$$

Dividing by the charge q we obtain the change in potential

$$\Delta V = \frac{\Delta U}{q} = -\vec{E} \cdot \Delta\vec{r}$$

Previously, we saw that for a uniform field (sketch a horizontal field and diagonal displacement)

$$\Delta V = -\vec{E} \cdot \Delta\vec{r} = -E\Delta r \cos\theta = -Ex$$

where $x = \Delta r \cos\theta$ is distance along a field line. Note that distance x can be positve or negative.

Consider the field of a point charge

$$\vec{E} = k\frac{Q}{r^2}\hat{r}$$

The field varies from point to along the displacement (**sketch**).

Consider a radial displacement from r_A to r_B.

Since the field is not constant along the path, to find the change in potential ΔV we must:

divide the displacement into an infinite number of segments, infinitesimal displacements $d\vec{r} = dr\,\vec{r}$,

evaluate $\vec{E}(\vec{r})$ for each segment, and

find the change in potential dV over each segment; since the field is constant over an infinitesimal segment we write simply

$$dV = -\vec{E} \cdot d\vec{r}$$

and perform an infinite sum from point A at $\vec{r}_A$ to point B at $\vec{r}_B$

$$\Delta V = \int_A^B -\vec{E}(\vec{r}) \cdot d\vec{r} = -\int_A^B E\,dr\;\vec{r} \cdot \vec{r} = -kQ\int_A^B \frac{1}{r^2}dr = -kQ\left[-\frac{1}{r}\right]_{r_A}^{r_B} = kQ\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

The integral is found in a table of integrals (or Wolfram's Integrator online).

The change in potential is independent of path, since each infinitesimal element of the path

$$d\vec{s} = dr\,\hat{r} + dl\,\hat{l} \qquad \textbf{\textit{sketch}}$$

has radial and lateral (tangential) components, and when dotted with $\vec{E} = E\,\vec{r}$ reduces to the preceding form

$$E\,\vec{r} \cdot d\vec{s} = E\left[dr\,\hat{r} \cdot \hat{r} + dl\,\hat{r} \cdot \hat{l}\right] = E\,dr, \qquad since\ \hat{r} \cdot \hat{r} = 1 \ \ and \quad \hat{r} \cdot \hat{l} = 0$$

recalling our dot products of unit vectors.

If we take the displacement to begin at infinity $r_A = \infty$, the potential difference is

$$V(r) = kQ\frac{1}{r_B} = k\frac{Q}{r}$$

If we choose our zero reference $V = 0$ at $r = \infty$, then the above formula gives the potential with respect to $r = \infty$.

We now have two expressions for the potential V, one for a uniform field

$$V = -Ex, \qquad where\ E\ points\ in + x\ direction$$

and one for the field of a point charge

$$V = k\frac{Q}{r}\,, \qquad where\ Q\ may\ be\ \pm$$

Both potentials can be $\pm$, the first since x can be $\pm$, the second since Q can be $\pm$ (r is always $+$).

Both potentials can be used to find the potential energy $U = qV$ of a charge q placed at x or r.

When computing V and U, remember: the zero references are at $x = 0$ and at $r = \infty$, respectively.

Example: the Hydrogen Atom. The potential created by the proton is

$$Q = e, \qquad V = k\frac{Q}{r} = k\frac{e}{r}$$

The potential energy of the electron at a distance r from the proton

$$q = -e, \qquad U = qV = -k\frac{e^2}{r}$$

is negative. If we plot $U(r)$ we see that the electron "lives" in a "potential well".

Suppose the electron in a hydrogen atom transitions from one orbit to another, from r_i to r_f

$$Q = e, \qquad q = -e, \qquad r_i = 2\text{Å}, \qquad r_f = 3\text{Å}$$

$$\Delta V = kQ\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = 2.4\ V$$

The change in potential energy of the electron is

$$\Delta U = q\Delta V = -e(2.4\,V) = -2.4\,eV = -3.84 \times 10^{-19}\,J$$

When making calculations involving electrons, it is customary to use electron volts eV, a convenient unit of energy.

You might think that the change in electron's kinetic energy is given by

$$\Delta K = -\Delta U$$

but something else happens: during the transition energy is radiated away in the form of light (take course PH1130 to learn about atomic transitions).

To find the change ΔK, we employ the equations of mechanics that describe circular motion.

The centripetal force exerted by the proton on the electron is related to kinetic energy K by

$$F = ma = \frac{mv^2}{r} = \frac{2}{r}\left(\frac{mv^2}{2}\right) = \frac{2}{r}K, \qquad K = \frac{mv^2}{2}$$

The force is also given by Coulomb's law

$$F = k\frac{e^2}{r^2}$$

Equating the two expressions for the force, we find

$$\frac{2}{r}K = k\frac{e^2}{r^2}\,, \qquad K = \frac{ke^2}{2r} = -\frac{U}{2}\,, \qquad since\ U = -k\frac{e^2}{r}$$

The change ΔK of the transition

$$\Delta K = K_f - K_i = -\frac{1}{2}\left(U_f - U_i\right) = -\frac{\Delta U}{2}$$

is half of the potential energy change; the other half of ΔU goes to photon energy.

The potential of a set of charges is simply the sum of the potentials of each, which can be derived using the preceding method (see Y&F Chap 23)

$$V = \sum_{i=1}^{n} k\frac{Q_i}{r_i} = k\left[\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \cdots\right]$$

where r_i is the distance between the field point to charge q_i (**sketch**).

We can plot the potentials $V(d)$ and $V(r)$ as functions of d and r, respectively.

We can also sketch the "equi-potential" surfaces. In 2D, these appear as lines and curves. In 3D the equipotentials are surfaces, planes and spheres.

The equi-potential lines are perpendicular to the electric field lines at points where the two cross. This can be shown graphically. Suppose the E and V lines cross at an angle other than 90°.

The E line indicates the direction of the E field. Break E field vector into components, one parallel to the V line. The parallel component $E_{\parallel}$ produces a change in potential along the V line, which contradicts the fact that V is constant along the V line.

Remember all the fun we had sketching the electric field lines? Now we can sketch the equi-potentials along with the field lines and obtain much more interesting patterns of lines (**use Caltech Applet**).

Rules for equi-potentials

The equi-potentials are perpendicular to the field lines.

The equi-potentials are more closely spaced in regions of stronger field.

Examples:

Given the field lines, sketch the equi-potentials.

Given the equi-potentials, sketch the field lines.

Given a set of charges and conductors, sketch the fields and equi-potentials.