

The Biot-Savart law

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0 I d\vec{L} \times \hat{r}}{4\pi r^2}$$

enables us to calculate the magnetic field produced by a current carrying wire of arbitrary shape.

We applied the law to determine the field of a long straight wire (length $L \gg r$)

$$B \cong \frac{\mu_0 I}{2\pi r}$$

at perpendicular distance r from the wire. The formula is exact for an infinitely long wire.

The field at points along the axis of a circular loop of wire of radius a is easily found (see Y&F figure 28.12). Referring to the figure, the magnitude of the field of a current element $I d\vec{L}$ is

$$|d\vec{L} \times \hat{r}| = dL \sin \varphi, \quad dB = \frac{\mu_0 I dL \sin \varphi}{4\pi r^2} = \frac{\mu_0 I dL}{4\pi r^2}, \quad r^2 = a^2 + x^2$$

where φ is the angle between $d\vec{L}$ and \hat{r} . From the figure we see that the angle is $\varphi = \pi/2$ for current elements all around the loop, so that $\sin \varphi = 1$.

The field $d\vec{B}$ of the element has a component parallel to the plane of the loop, and another component parallel to the axis of the loop (the x axis in the figure). The axial component points in the $+x$ direction and has magnitude

$$dB_x = dB \cos \theta = \frac{\mu_0 I dL a}{4\pi r^2 r} = \frac{\mu_0 I a dL}{4\pi r^3}, \quad \cos \theta = \frac{a}{r}$$

We will not bother with the planar component. Why? Because each current element is paired with a current element on the opposite side of the loop, and their planar components cancel.

We may write the infinitesimal length dL in terms of an infinitesimal angle $d\alpha$

$$dL = a d\alpha$$

The field at the point x along the axis is found by integrating over length around the loop, or equivalently, integrating over angle around the loop $0 \leq \alpha \leq 2\pi$

$$\vec{B} = \frac{\mu_0 I a}{4\pi r^3} \hat{x} \oint_L dL = \frac{\mu_0 I a}{4\pi r^3} \hat{x} \int_0^{2\pi} a d\alpha = \frac{\mu_0 I a^2}{2r^2} \hat{x} = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \hat{x}$$

Notice how everything except dL is a constant and may be brought outside the integral (the infinite sum). The integral over the angle yields a factor of 2π .

For a coil of N turns of wire the field at a point x on the axis of the coil is

$$\vec{B} = \frac{\mu_0 I a^2 N}{2(x^2 + a^2)^{3/2}} \hat{x}$$

The field at the center of the coil is obtained by setting $x = 0$

$$\vec{B} = \frac{\mu_0 I a^2 N}{2(0 + a^2)^{3/2}} \hat{x} = \frac{\mu_0 N I}{2a} \hat{x}, \quad B = \frac{\mu_0 N I}{2a}$$

Curl your right hand fingers in the direction of the current, then your thumb points in the \hat{x} direction.

For locations in space that are off-axis, the field lines form lop-sided ovals around the wire (similar to the circular field lines around a straight wire).

Ampere's Law. Let's review the field of a long straight wire. The field lines form circles around the wire

$$B = \frac{\mu_0 I}{2\pi r}$$

The field has constant magnitude along a circle of radius r , centered on the wire.

Writing the formula in a slightly different way

$$B(2\pi r) = \mu_0 I$$

we see that the product of the field strength B multiplied by the circumference of the circle is equal to the encircled current I multiplied by the permeability constant. This is true for any value of r .

When we say that that the circle encloses the wire, we mean that the wire passes through the circle.

You could, if you wanted, write the formula as the infinite sum (the integral) of elements $B dL$, where dL is an infinitesimal piece of the circumference.

$$\oint B dL = \mu_0 I_{encl}$$

Ampere came up with the bright idea that this more general relationship holds true not only for a circle, but for any closed path that encloses the current, and for any magnet field, uniform or not, provided you use the component of the field parallel to dL at each point along the path (for the circle above, the field is tangent to the circle at all points)

$$\oint \vec{B} \cdot d\vec{L} = \oint B dL \cos \theta = \mu_0 I_{encl}, \quad \text{Ampere's Law}$$

In words, Ampere's law states that the "circulation" of the field equals the enclosed current times μ_0 .

Did you notice that there are two ways you can point $d\vec{L}$, that is, two ways you can "walk" around the loop, clockwise or counter-clockwise? Which way should we go to find the circulation? Does it matter? Yes, it does matter; the direction determines the sign, + or - of the circulation. To establish our

convention, let's refer to the simplest case, a long straight wire. Point your thumb in the direction of the current; your fingers curl in the direction of the field lines. If you "walk" in the direction indicated by your fingers, the circulation is positive, and the current that flows in the direction of your thumb is a positive current.

Stated one more way: if you walk counter-clockwise around an Amperian loop that lies in the plane of the page, a positive enclosed current points out of the page and will produce a magnetic field that results in a positive circulation.

Let's apply Ampere's Law to a long solenoid to find the field inside the solenoid.

A solenoid is formed by wrapping wire around a tube with the windings closely spaced.

Applying the Biot-Savart law to this helical wire reveals that for a long, tightly wound solenoid, the field is very strong and very uniform inside the tube, and very weak outside the tube.

Let's consider an ideal solenoid (infinitely long and no space between the windings, for which field is zero outside the solenoid and of constant magnitude inside the solenoid).

For a length of solenoid L containing N windings each carrying current I , and for an "Amperian" rectangular loop that encloses N windings (see Y&F figure 28.23 page 975),

Ampere's law gives

$$\oint \vec{B} \cdot d\vec{L} = \int_{in} \vec{B}_{in} \cdot d\vec{L} + \int_{out} \vec{B}_{out} \cdot d\vec{L} = B_{in}L + 0 = \mu_0 I_{encl} = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{L} = \mu_0 nI, \quad n = \frac{N}{L}$$

where n is the number density of turns (turns per meter length of solenoid).

At last, we have learned how to create a uniform magnetic field! You can put your solenoid on the shelf next to your parallel plate capacitor. Remember all those times we made use of a uniform magnetic field? Remember the mass spectrometer?

Examples

Two wires carry currents I_1 and I_2 , one current is into the page, the other out of the page. What is the magnetic field circulation around a loop that encloses the wires?

$$circulation = \oint \vec{B} \cdot d\vec{L} = \mu_0 I_{encl} = \mu_0 (I_1 + I_2)$$

What is the circulation around a loop that does not enclose any of the wires?

$$circulation = \oint \vec{B} \cdot d\vec{L} = \mu_0 I_{encl} = 0$$