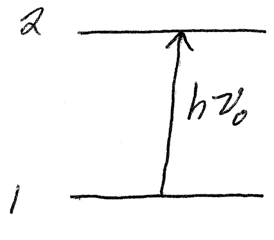
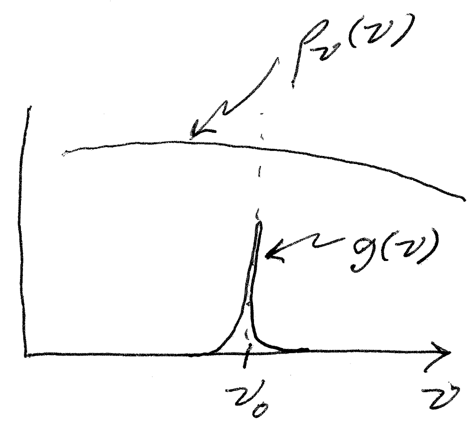
Transition Rates for Monochromatic Radiation

Einstein's derivation:

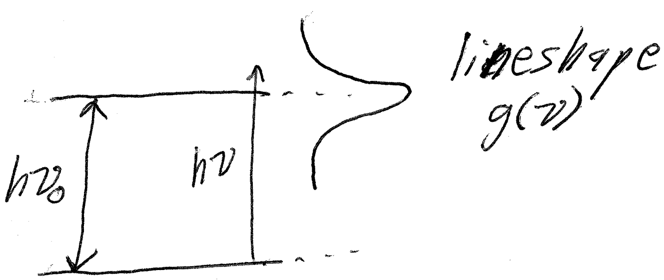


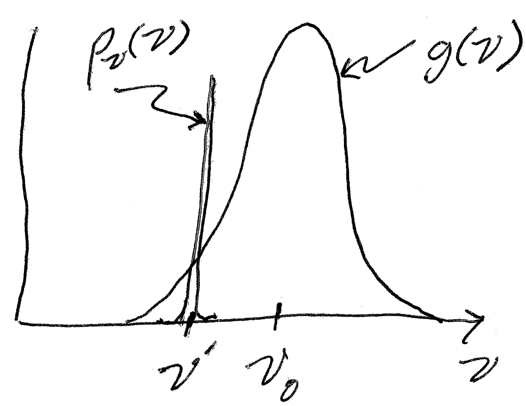
$$W_{12}^{ind} = B_{12}\,\rho_\nu(\nu_0)$$



Must generalize

$$W_{12} = \int B_{12}\,\rho_\nu(\nu)\,g(\nu)\,d\nu$$





For $\rho_\nu(\nu)$ much narrower than $g(\nu)$,

$$W_{12} \simeq B_{12}\,g(\nu') \int \rho_\nu(\nu)\,d\nu$$

$$\boxed{W_{12} \simeq B_{12}\,g(\nu')\,\rho} \qquad \rho \equiv \int \rho_\nu(\nu)\,d\nu$$

Now $\rho = \dfrac{\text{total energy}}{\text{Vol}}$

before $\rho_\nu = \dfrac{\text{energy}}{\Delta\nu \cdot \text{Vol}}$

Other limit reduces to Einstein case:

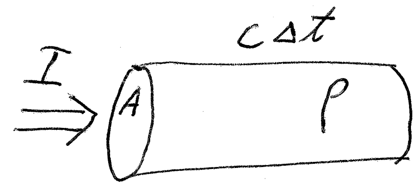
$$W_{12} \approx B_{12}\, \rho_\nu(\nu_0) \underbrace{\int g(\nu)\, d\nu}_{\approx 1 \text{ normalized}}$$

Intensity

$$I = c\,\rho$$

I: $\dfrac{\text{energy}}{\text{area} \cdot \text{time}}$

ρ: $\dfrac{\text{energy}}{\text{Vol.}}$

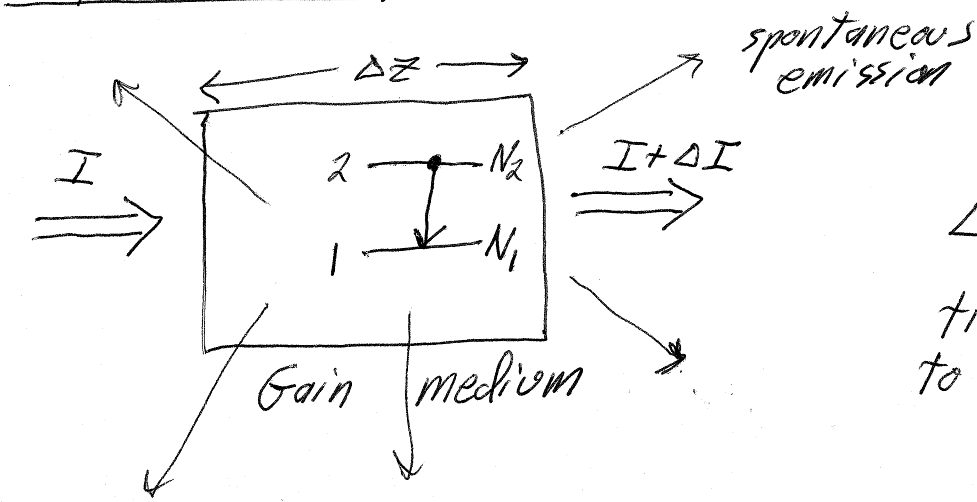


$$I A \Delta t = \rho A c \Delta t$$

So

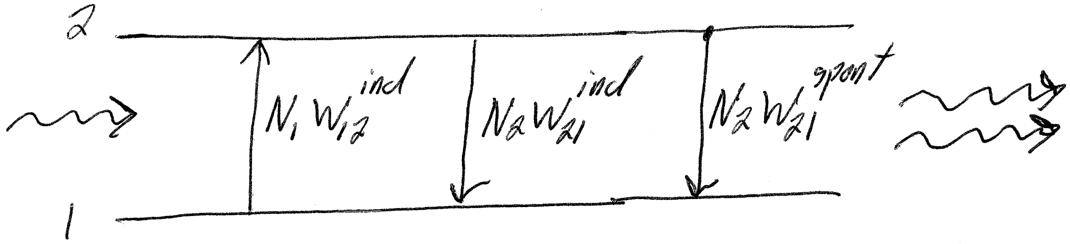
$$\boxed{W_{12} = B_{12}\, g(\nu')\, \frac{I}{c}}$$

Amplification by Atomic System



$$\Delta t = \frac{\Delta z}{c}$$

time for light to pass through Δz



$$\frac{\Delta\rho}{\Delta t} = \frac{\text{energy generated}}{(\text{unit time})(\text{Vol})} = h\nu\left(N_2 W_{21}^{ind} - N_1 W_{12}^{ind}\right)$$

But since $B_{12} = B_{21}$, then $W_{12}^{ind} = W_{21}^{ind}$

so

$$\frac{\Delta\rho}{\Delta t} = h\nu\, W_{21}^{ind}\,(N_2 - N_1)$$

$$\frac{\Delta\rho}{\Delta t} = h\nu\, W_{21}^{ind}\,\Delta N$$

$\Delta N \equiv N_2 - N_1$
"population inversion"

$$\Delta I = c\,\Delta\rho = h\nu\, W_{21}^{ind}\,\Delta N\,\Delta z$$

Now use

$$W_{21}^{ind} = B_{21}\, g(\nu')\,\frac{I}{c}$$

$$= A_{21}\,\frac{c^3}{8\pi h\nu^3}\, g(\nu')\,\frac{I}{c}$$

$$= A_{21}\,\frac{\lambda^2}{8\pi}\, g(\nu')\, I\,\frac{1}{h\nu}$$

To obtain

$$\Delta I = A_{21}\,\frac{\lambda^2}{8\pi}\, g(\nu')\, I\,\Delta N\,\Delta z$$

Take limit as $\Delta z \to 0$

$$\boxed{\frac{dI}{dz} = \gamma I} \qquad \boxed{\gamma(\nu) = A_{21}\,\frac{\lambda^2}{8\pi}\, g(\nu)\,\Delta N}$$

gain coefficient

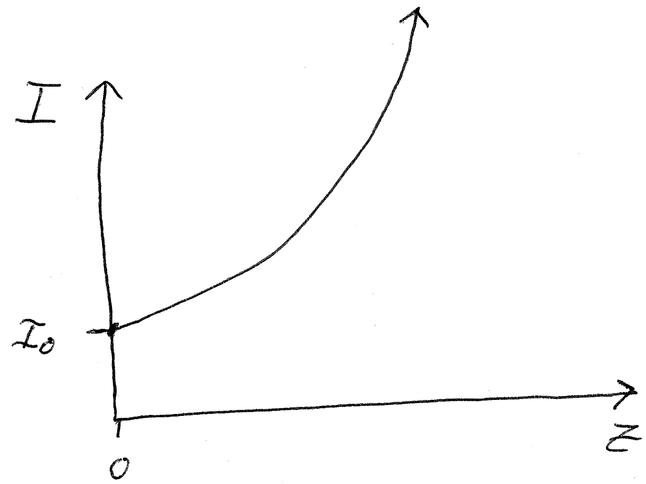
In gain medium with refractive index n,

$\lambda \to \lambda/n$ so

$$\boxed{\gamma(\nu) = A_{21} \frac{\lambda^2}{8\pi n^2} g(\nu) \Delta N}$$

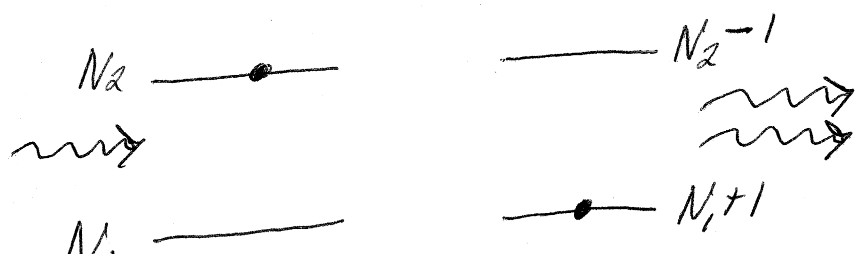
If γ = constant, solution for $I(z)$ is

$$I(z) = I(0) e^{\gamma z}$$



goes to ∞: not realistic

Actually, γ not constant because ΔN not constant



before

$\Delta N = N_2 - N_1$

after

$\Delta N' = (N_2 - 1) - (N_1 + 1)$
$= \Delta N - 2$

This is called "saturation" of the gain

Absorption and Emission Cross Section

$$\gamma(\nu) \cong \Delta N \, \sigma(\nu) = (N_2 - N_1)\,\sigma(\nu)$$

gain coefficient = (population difference)(cross section)

$$\boxed{\sigma(\nu) = A_{21} \frac{\lambda^2}{8\pi n^2} g(\nu)}$$

depends on properties of a single atom

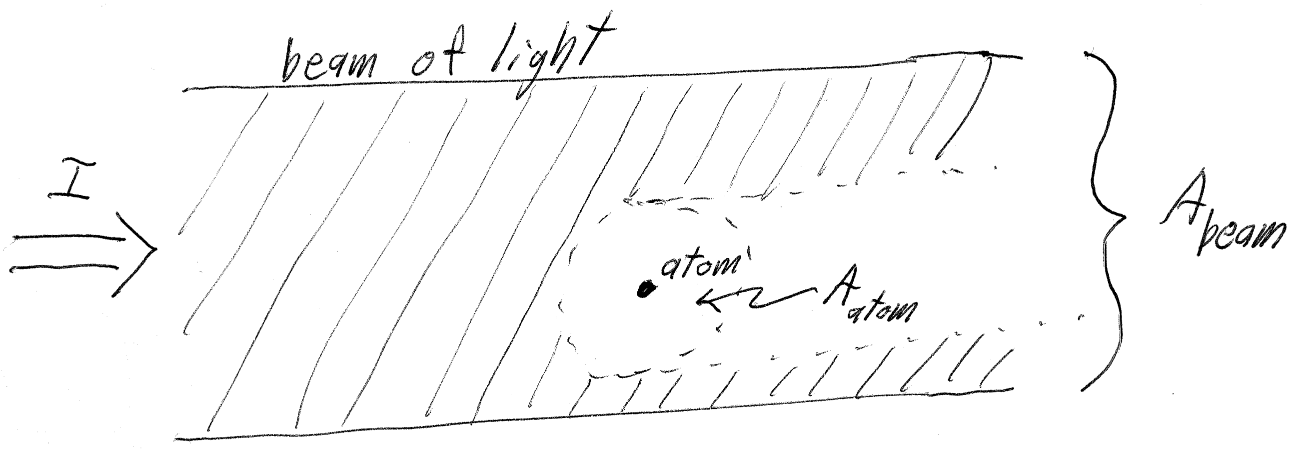
This cross section can be interpreted as the effective area within which an atom absorbs or emits light.

Consider $N_2 = 0$ (no excited atoms)

$$\gamma = -N_1 \sigma$$

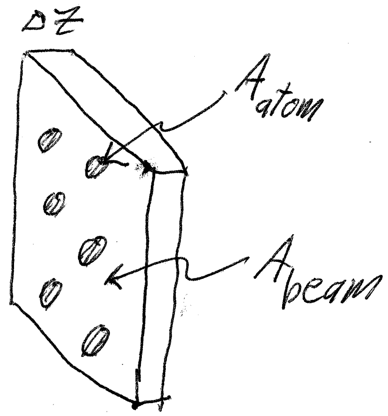
Define $\alpha \equiv N_1 \sigma$ absorption coefficient

Then $I(z) = I(0)\, e^{-\alpha z}$ (Beer's law)



Each atom absorbs light within an area A_{atom}

Fraction of light absorbed by each atom is $\left(\frac{A_{atom}}{A_{beam}}\right)$



\# atoms in Δz is

$$= N_1 A_{beam} \Delta z$$

Fraction of light absorbed in Δz is

$$frac = (N_1 A_{beam} \Delta z)\left(\frac{A_{atom}}{A_{beam}}\right) = N_1 A_{atom} \Delta z$$

But from definition of gain coefficient,

$$\frac{\Delta I}{I} = -\alpha \Delta z \qquad \text{so} \qquad frac = \alpha \Delta z$$

Therefore,

$$\alpha \Delta z = N_1 A_{atom} \Delta z$$

$$N_1 \sigma = N_1 A_{atom}$$

$$\boxed{\sigma = A_{atom}}$$

Cross section $\sigma(\nu)$:

* property of single atom
* varies with ν or λ [lineshape function $g(\nu)$]
* can look up $\sigma(\nu)$ data in handbooks
* same shape as lineshape function $g(\nu)$