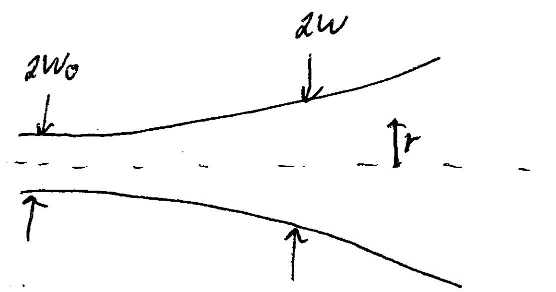
Intensity of Gaussian Beam

$$E(r) = E_0 \frac{w_0}{w} e^{-\frac{r^2}{w^2}}$$



Energy density

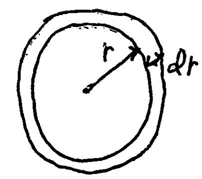
$$\rho = \tfrac{1}{2}\epsilon_0 E^2$$

Intensity $\quad I = c\rho = \tfrac{1}{2} c \epsilon_0 E^2$

E is peak field

Power in beam

$$P = \int I\, dA = \int_0^\infty I\, 2\pi r\, dr$$



$$P = 2\pi \tfrac{1}{2} c \epsilon_0 \int_0^\infty E^2 r\, dr$$

$dA = 2\pi r\, dr$

$$P = \pi c \epsilon_0 E_0^2 \frac{w_0^2}{w^2} \int_0^\infty e^{-\frac{2r^2}{w^2}} r\, dr$$

Let $u \equiv -\frac{2r^2}{w^2} \qquad du = -2\cdot\frac{2r}{w^2}\, dr$

$$\int_0^{-\infty} e^{+u}\, du = -1 \qquad e^{-\infty} - e^{0}$$

$$P = \frac{\pi}{2} w_0^2 \tfrac{1}{2} c \epsilon_0 E_0^2 \quad \text{note: } P \text{ independent of } w \text{ (and } z\text{)}$$

$$P = I_{max} A_{o,eff}$$

where $\boxed{A_{o,eff} = \tfrac{1}{2}\pi w_0^2 \qquad I_{max} = \tfrac{1}{2} c \epsilon_0 E_0^2}$

at beam waist

Intensity on axis, not at beam waist:

$$I_{max} = \frac{P}{A_{eff}} = \frac{P}{\tfrac{1}{2}\pi w^2}$$

define $A_{eff} \equiv \tfrac{1}{2}\pi w^2$ arbitrary z

$$E^2 w^2 = E_0^2 w_0^2$$

$$I w^2 = I_0 w_0^2$$

so $P = I_{axis} A_{eff}$ any z

Interaction of Light with Matter

Einstein A & B coefficients

$$\rho_\nu(\nu) = \frac{1}{V}\frac{dN}{d\nu} = \frac{8\pi\nu^2}{c^3} \qquad \left[\frac{\#\text{modes}}{\Delta\nu\,\Delta V}\right]$$

Planck distribution:

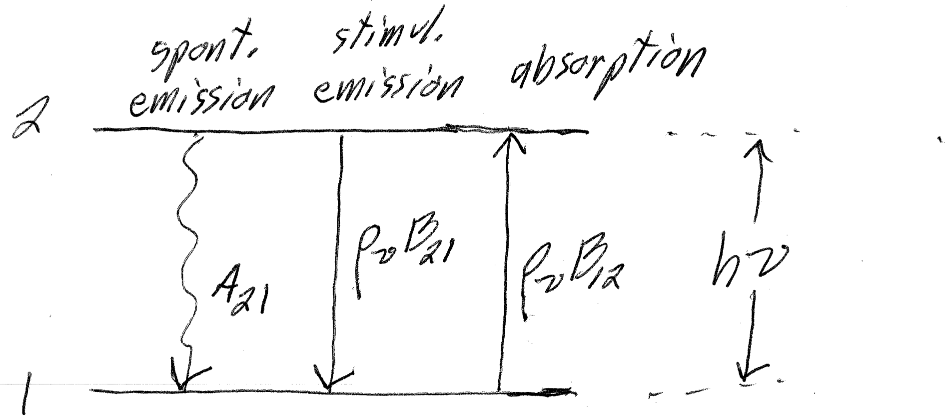
$$\rho_\nu(\nu) = \frac{8\pi\nu^2}{c^3}\,\frac{h\nu}{e^{\frac{h\nu}{kT}}-1} \qquad \left[\frac{\text{energy}}{\Delta\nu\,\Delta V}\right]$$

and

$$h\nu = \frac{\text{energy}}{\text{photon}}$$

$$\underbrace{\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}}_{\frac{\text{energy}}{\text{mode}}} = \underbrace{h\nu}_{\frac{\text{energy}}{\text{photon}}} \cdot \underbrace{\frac{1}{e^{\frac{h\nu}{kT}}-1}}_{\frac{\#\text{photons}}{\text{mode}}}$$

* This anticipates quantum statistics (bosons)
* Einstein could derive Planck distribution by hypothesizing stimulated emission (1917)



rates
$$W_{21}^{spont} = A_{21} = \frac{\text{prob. that atom emits spontaneously}}{\text{unit time}}$$
$$W_{21}^{ind} = \rho_\nu B_{21} = \frac{\text{prob}}{\text{time}} \text{ for induced emission}$$
$$W_{12}^{ind} = \rho_\nu B_{12} = \frac{\text{prob}}{\text{time}} \text{ for absorption}$$

level populations
$$N_2 = \frac{\#\text{ atoms in level 2}}{\text{Vol}}$$
$$N_1 = \frac{\#\text{ atoms in level 1}}{\text{Vol}}$$

rate equation
$$\frac{dN_2}{dt} = -N_2 A_{21} - N_2 \rho_\nu B_{21} + N_1 \rho_\nu B_{12}$$

In equilibrium, $\frac{dN_2}{dt} = 0$

so
$$\frac{N_2}{N_1} = \underbrace{\frac{B_{12}\rho_\nu}{A_{21} + B_{21}\rho_\nu}}_{\text{from rate eq.}} = \underbrace{e^{-\frac{h\nu}{kT}}}_{\text{Boltzmann factor}}$$

evaluate at high temp where $\rho_\nu \propto kT$

Note that A & B independent of temp

As $T \to \infty$, $\frac{N_2}{N_1} \to \frac{B_{12}}{B_{21}} = 1$

$\therefore$ $\boxed{B_{12} \approx B_{21}}$ Can drop subscript on B

Now back to finite temp:

$$e^{-\frac{h\nu}{kT}} = \frac{B\rho_\nu}{A + B\rho_\nu}$$

Solve for ρ_ν:

$$\rho_\nu(\nu) = \frac{A}{B}\frac{e^{-\frac{h\nu}{kT}}}{1 - e^{-\frac{h\nu}{kT}}} = \frac{A}{B}\frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

Comparing with BB spectrum,

$$\boxed{\frac{A_{21}}{B_{21}} = \frac{8\pi\nu^2}{c^3}\cdot h\nu} = \rho_0\, h\nu$$

Use $\rho_\nu(\nu) = \left(\frac{\text{modes}}{\text{Vol}\ \Delta\nu}\right)\left(\frac{\text{photons}}{\text{mode}}\right)\left(\frac{\text{energy}}{\text{photon}}\right)$

$= \rho_\nu(\nu)\ \bar{n}\ h\nu$

To obtain $W_{21}^{ind} = \rho_\nu(\nu) B_{21} = \left[\rho_\nu(\nu) h\nu B_{21}\right]\bar{n}$

Then $$\frac{W_{21}^{spont}}{W_{21}^{ind}} = \frac{B_{21}\,\rho_\nu\, h\nu}{\rho_\nu\, h\nu\, B_{21}\,\bar{n}} = \frac{1}{\bar{n}}$$

$$\boxed{W_{21}^{ind} = \bar{n}\ W_{21}^{spont}}$$

$\bar{n} = \frac{1}{e^{\frac{h\nu}{kT}} - 1}$ $\left[\frac{\text{photons}}{\text{mode}}\right]$