Two variables that are inter-related:

N_2 excited state population

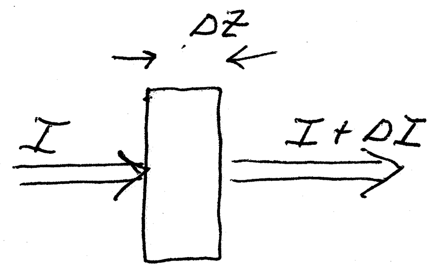
I light intensity inside cavity

We have one equation relating them:

$$\boxed{\frac{dN_2}{dt} = R - N_2\left(\frac{I\sigma}{h\nu} + \frac{1}{\tau_2}\right)} \qquad (1)$$

Need another equation.

Consider light propagating distance Δz in time $\Delta t = \Delta z / c$



$$\frac{\Delta I}{I} = (\gamma - \gamma_{th})\,\Delta z$$

$$\frac{\Delta I}{I} = (\gamma - \gamma_{th})\,c\,\Delta t$$

$$\gamma = \sigma N_2$$

$$\frac{\Delta I}{\Delta t} = c(\gamma - \gamma_{th}) I$$

$$= (c\sigma N_2 - c\gamma_{th}) I$$

$$\boxed{\frac{dI}{dt} = c\sigma N_2 I - \frac{I}{\tau_c}} \qquad (2)$$

where

$$\tau_c \equiv \frac{1}{c\gamma_{th}} = \frac{1}{c\alpha + \frac{c}{2L}\ln\left(\frac{1}{R_1 R_2}\right)}$$

τ_c : cavity lifetime, similar to photon lifetime defined earlier.

Eqs. (1) & (2) are coupled, nonlinear diff. eqs.

Note: if $I(0) \approx 0$, $I(t) \approx 0$ all t

We have neglected spontaneous emission in (2)

For lasing, $\frac{dI}{dt} > 0$, or

$$c\sigma N_2 \geq \frac{1}{\tau_c}$$

$$\boxed{N_{2,th} = \frac{1}{c\sigma\tau_c}}$$

Can re-write using $\quad \frac{1}{c\tau_c} = \gamma_{th}$

$$\sigma \approx A_{21} \frac{\lambda^2}{8\pi} g(\nu)$$

$$\boxed{N_{2,th} = \frac{8\pi\gamma_{th}}{A_{21}\lambda^2 g(\nu)}}$$

same result as before

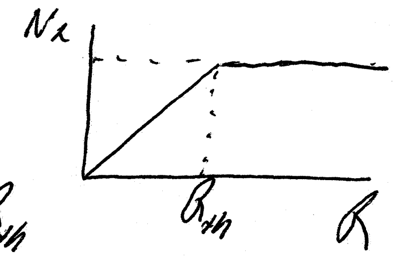
Steady State Solutions

$$\frac{dN_2}{dt} = 0 \qquad\qquad \mathcal{R} = N_2\left(\frac{I\sigma}{h\nu} + \frac{1}{\tau_2}\right)$$

(1) Below Threshold

I very small (no lasing) so

$$\mathcal{R} \cong \frac{N_2}{\tau_2}$$



As increase $\mathcal{R}$, N_2 increases up to $N_{2,th}$ at $\mathcal{R}_{th}$

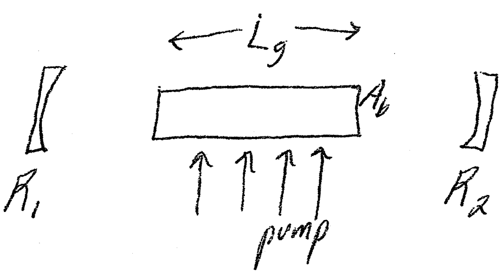
$$\mathcal{R}_{th} = \frac{N_{2,th}}{\tau_2} = \frac{1}{c\sigma\tau_c\tau_2}$$

Pump power required to reach threshold:

$$P_{th} = \mathcal{R}_{th}\, V\, h\nu_p$$

Example:

Find pump power to achieve lasing in Nd: YAG rod



pump at $\langle \lambda_p \rangle = 500$ nm with lamp

Take $R_1 = 1$ $\quad R_2 \simeq 0.85$

$L_g = 7.5$ cm $\quad A_b = 0.23\ cm^2$

$\sigma_p \simeq 2.8 \cdot 10^{-19}\ cm^2$ $\quad \lambda = 1.064\ \mu m$

$\tau_{21} = 230\ \mu s$ $\quad \alpha = 5 \cdot 10^{-3}\ cm^{-1}$

Then

$$\frac{1}{c\tau_c} = \gamma_{th} = \alpha + \frac{1}{2L_g} \ln\left(\frac{1}{R_1 R_2}\right) = 0.0158\ cm^{-1}$$

$$R_{th} = \frac{\gamma_{th}}{\sigma \tau_2} = \frac{(.0158\ cm^{-1})}{(2.8 \cdot 10^{-19}\ cm^2)(2.3 \cdot 10^{-4}\ s)} = 2.45 \cdot 10^{20} \frac{1}{cm^3 s}$$

$$h\nu_p = \frac{hc}{\lambda_p} = \frac{(6.63 \cdot 10^{-34})(3 \cdot 10^{8})}{500 \cdot 10^{-9}} = 3.98 \cdot 10^{-19}\ J$$

$$P_{th} = R_{th} V h\nu_p = \left(2.45 \cdot 10^{20} \frac{1}{cm^3 s}\right)(0.23\ cm^2)(7.5\ cm)(3.98 \cdot 10^{-19}\ J)$$

$P_{th} = 168$ W (threshold for absorbed optical power)

Taking $\eta_p \equiv \dfrac{P_{abs}\text{(absorbed optical)}}{P_{elec}\text{ (electrical power to lamp)}} = 0.05$

$$P_{th}^{elec} = \frac{1}{\eta_p} P_{th}^{abs} = \frac{168\ W}{0.05} = 3.36\ kW$$

(2) Above Threshold

After lasing starts, and reaches steady-state, must have

$$\frac{dI}{dt} = \left(c\sigma N_2 - \frac{1}{\tau_c}\right) I = 0$$

$$N_2 = \frac{1}{c\sigma\tau_c} = N_{2,th}$$

or, N_2 becomes "pinned" at its threshold value.

(otherwise, I would increase exponentially)

Putting $N_2 \approx N_{2,th}$ into eq. (1) and solving for I gives

$$\frac{dN_2}{dt} = R - N_{2,th}\left(\frac{I\sigma}{h\nu} + \frac{1}{\tau_2}\right) = 0$$

$$R = R_{th}\left(\frac{I\sigma\tau_2}{h\nu} + 1\right)$$

$$\boxed{I = \frac{h\nu}{\sigma\tau_2}\left[\frac{R}{R_{th}} - 1\right]}$$

where $R_{th} = \frac{N_{2,th}}{\tau_2}$