Homework #3

18.2 (a) Using the maximum value of $g(\nu)$ from Eq.(18-50), $g_{max} = 2/(\pi\,\Delta\nu)$, we have from Eq.(18-36)

$$\sigma_{peak} = A_{21}\frac{\lambda^2}{8\pi n^2}g_{max} = A_{21}\frac{\lambda^2}{8\pi n^2}\frac{2}{\pi\,\Delta\nu} = \frac{\lambda^2}{\tau_{21}4\pi^2 n^2\,\Delta\nu}$$

where we have replaced $\lambda \rightarrow \lambda/n$ for a medium with refractive index n, and have used $A_{21} = 1/\tau_{21}$.

(b) From Eq.(18-52), the Gaussian lineshape function has a peak value of

$$g_{max} = \frac{2}{\Delta\nu}\sqrt{\frac{\ln 2}{\pi}}$$

so

$$\sigma_{peak} = \frac{\lambda^2}{\tau_{21}8\pi n^2}\frac{2}{\Delta\nu}\sqrt{\frac{\ln 2}{\pi}} = \frac{\lambda^2}{\tau_{21}4\pi n^2\,\Delta\nu}\sqrt{\frac{\ln 2}{\pi}}$$

18.3 (a) Using the results of problem 18.2,

$$\tau_{21} = \frac{\lambda^2}{\sigma_{peak}4\pi^2 n^2\,\Delta\nu} = \frac{(694\times 10^{-9})^2}{(2.5\times 10^{-24})4\pi^2(1.76)^2(3.3\times 10^{11})} = 4.8\times 10^{-3}\text{ s} = 4.8\text{ ms}$$

(b) The quantum efficiency is $\phi = \tau/\tau_r = 3/4.8 = 0.63$.

(c) The radiative rate is $W_r = 1/\tau_r = 1/(4.8\times 10^{-3}) = 208\text{ s}^{-1}$, and the total decay rate is $W_{tot} = 1/\tau = 1/(3\times 10^{-3}) = 333\text{ s}^{-1}$. The nonradiative rate is therefore $W_{nr} = 333 - 208 = 125\text{ s}^{-1}$.

18.5 (a) Using the results of problem 18.3 we have $A_{21} = 1/\tau_{21} = 1/(4.8 \times 10^{-3}) = 208\ \text{s}^{-1}$. The Einstein B coefficient is evaluated from Eq.(18-10) (using $c \rightarrow c/n$ for index n) as

$$B_{21} = \frac{A_{21}c^3}{8\pi n^3 h\nu^3} = \frac{A_{21}\lambda^3}{8\pi n^3 h} = \frac{(208)(694 \times 10^{-9})^3}{8\pi(1.76)^3(6.63 \times 10^{-34})} = 7.65 \times 10^{14}\ \frac{\text{m}^3}{\text{J} \cdot \text{s}^2}$$

(b) Using Eq.(18-20) with

$$g_{max} = \frac{2}{\pi\ \Delta\nu} = \frac{2}{\pi(3.3 \times 10^{11})} = 1.93 \times 10^{-12}\ \text{s}$$

the induced transition rate is

$$W_{21}^{ind} = B_{21}g_{max}\frac{I}{c/n} = \left(7.65 \times 10^{14}\ \frac{\text{m}^3}{\text{J} \cdot \text{s}^2}\right)\left(1.93 \times 10^{-12}\ \text{s}\right)\left(10^{13}\ \frac{\text{W}}{\text{m}^2}\right)\frac{1.76}{3 \times 10^8\ \text{m/s}}$$

or

$$W_{21}^{ind} = 8.6 \times 10^7\ \text{s}^{-1}$$

Under these conditions, $W_{21}^{ind} \gg A_{21}$, so induced emission dominates the decay of the excited state.

18.6 (a) From Fig. 18-10, the absorption cross section at 1480 nm is $\sigma_{abs}(1480) \approx 1.3 \times 10^{-21}$ cm^2. The absorption coefficient is then

$$\alpha = N\sigma = (8 \times 10^{18}\ \text{cm}^{-3})(1.3 \times 10^{-21}\ \text{cm}^2) = 1.04 \times 10^{-2}\ \text{cm}^{-1}$$

(b) The fraction of pump light absorbed is

$$\text{frac} = 1 - e^{-\alpha L} = 1 - e^{-(0.0104)(200)} = 0.875$$

(c) With $N_2 \simeq 8 \times 10^{18}$ cm^{-3} and $N_1 \ll N_2$, the gain coefficient becomes

$$\gamma \simeq N_2\sigma_{em}(1560) = (8 \times 10^{18}\ \text{cm}^{-3})(1.7 \times 10^{-21}\ \text{cm}^2) = 0.0136\ \text{cm}^{-1}$$

where $\sigma_{em}(1560)$ is estimated from Fig. 18-10.

(d) The gain in a length of 2 m (200 cm) is

$$G = \frac{I}{I_0} = e^{\gamma L} = e^{(0.0136)(200)} = 15.2$$

18.10 (a) Beer's law gives $0.6 = e^{-\alpha L}$, with $L = 10$ cm, and solving for α yields

$$\alpha = \frac{1}{L} \ln\left(\frac{1}{0.6}\right) = \frac{1}{10\text{ cm}} \ln\left(\frac{1}{0.6}\right) = 5.11 \times 10^{-2}\text{ cm}^{-1}$$

(b) In a length of 25 cm the fraction transmitted would be

$$T = e^{-\alpha L} = e^{-(0.0511)(25)} = 0.279$$

18.11 (a) Since the area under a symmetric triangle is 1/2 the base times the height, the area under the $g(\nu)$ lineshape function would be (1/2)(base)g_{max}. But for a symmetric triangle, the full width at half maximum ($\Delta\nu$) is 1/2 the base, and so the area is just $g_{max}\,\Delta\nu$. Since the area under $g(\nu)$ is 1 by definition, we have the simple result for the triangle distribution that $g_{max} = 1/\Delta\nu$. The frequency width is calculated from the wavelength width by

$$\Delta\nu = \frac{c}{\lambda^2}\,\Delta\lambda = \frac{(3\times10^8)(40\times10^{-9})}{(590\times10^{-9})^2} = 3.45\times10^{13}\text{ Hz}$$

The peak cross section is then given by Eq.(18-36) as

$$\sigma_{peak} = \frac{\lambda^2}{8\pi n^2 \tau_{21}\,\Delta\nu} = \frac{(590\times10^{-9})^2}{8\pi(1.33)^2(8\times10^{-9})(3.45\times10^{13})} = 2.84\times10^{-20}\text{ m}^2$$

(b) The peak gain coefficient is

$$\gamma_{peak} = \Delta N\,\sigma_{peak} = \left(10^{18}\text{ cm}^{-3}\right)\left(2.84\times10^{-16}\text{ cm}^2\right) = 284\text{ cm}^{-1}$$

(c) We require that $\exp(\gamma L) = 20$, so

$$L = \frac{1}{\gamma}\ln 20 = \frac{1}{284}\ln 20 = 0.0105\text{ cm}$$

19.2 (a) The excitation rate is

$$\mathcal{R} = N_0 W_p = \left(3 \times 10^{20}\ \mathrm{cm}^{-3}\right)\left(170\ \mathrm{s}^{-1}\right) = 5.1 \times 10^{22}\ \mathrm{cm}^{-3}\ \mathrm{s}^{-1}$$

(b) Eq.(19-7) gives

$$N_2 = \mathcal{R}\tau_2 = (5.1 \times 10^{22})(290 \times 10^{-6}) = 1.48 \times 10^{19}\ \mathrm{cm}^{-3}$$

(c) The fraction of ions in the excited state is $N_2/N_0 = 1.48/30 = 0.049$

(d) The gain coefficient is

$$\gamma = N_2\sigma = \left(1.48 \times 10^{19}\ \mathrm{cm}^{-3}\right)\left(4 \times 10^{-20}\ \mathrm{cm}^2\right) = 0.59\ \mathrm{cm}^{-1}$$

19.3 (a) The sponteneous emission rate is the reciprical of the spontaneous lifetime, or $1/\tau_2 = 1/(290 \times 10^{-6}) = 3.45 \times 10^3\ \mathrm{s}^{-1}$. The induced emission rate is

$$W_{21}^{ind} = \frac{I\sigma}{h\nu} = \frac{I\sigma\lambda}{hc} = \frac{(5 \times 10^8)(4 \times 10^{-24})(1054 \times 10^{-9})}{(6.63 \times 10^{-34})(3 \times 10^8)} = 1.06 \times 10^4\ \mathrm{s}^{-1}$$

(b) Using the unsaturated value of N_2 obtained in problem 19.2, we have

$$N_2 = \frac{\mathcal{R}\tau_2}{1 + W_{21}^{ind}\tau_2} = \frac{1.48 \times 10^{19}\ \mathrm{cm}^{-3}}{1 + (1.06 \times 10^4)(290 \times 10^{-6})} = 3.63 \times 10^{18}\ \mathrm{cm}^{-3}$$

(c) The effective lifetime is τ_2', given by

$$\frac{1}{\tau_2'} = \frac{1}{\tau_2} + W_{21}^{ind} = 3.45 \times 10^3 + 10.6 \times 10^3 = 1.405 \times 10^4\ \mathrm{s}^{-1}$$

$$\tau_2' = 7.1 \times 10^{-5}\ \mathrm{s} = 71\,\mu\mathrm{s}$$

(d) The gain coefficient is

$$\gamma = N_2\sigma = \left(3.63 \times 10^{18}\ \mathrm{cm}^{-3}\right)\left(4 \times 10^{-20}\ \mathrm{cm}^2\right) = 0.145\ \mathrm{cm}^{-1}$$