Physics 1111

Springs

Springs are one of my less favorite topics in freshman physics. It is very easy to reduce a discussion of springs to a single incorrect statement of Hooke's law. It is even easier if I only give you cooked problems that hide the fact that you actually don't know what you are doing [that would be a teaching failure, not a learning failure.] The conceptual issues are straightforward. The challenge arises because getting correct answers requires that you be meticulous in assigning signs to algebraic characters. Let's look at several topics in sequence.

Masslessness

Massless refers to the approximation that the mass of an object is negligibly small relative to the mass of the other objects in the system. A mass is negligible if it is not important to calculated answers. That is, a mass is negligible in a particular calculation if you can change the numerical value of the mass, and the outcome of the calculation does not change by an amount that you care about.

Aside: "Negligible" does not tell you how small a mass must be to be negligible. Why are there complications? The core issue is that in some mechanics problems the outcome depends on an input mass as a polynomial in m, so that a 10% change in a mass typically changes the calculated answer by some number like 10%. However, some other mechanics problems are exponentially sensitive to input parameters, and have answers that diverge exponentially quickly from each other as input parameters are also changed. In these problems, a 10% change in a mass has an astronomically huge effect. In this course, we mostly stay away from the "some other" problems. Planetary astronomy and billiards played with ideal frictionless billiard balls are good examples of the "some other" problems.

Let us consider what happens when we apply a set of forces to an object whose mass becomes negligibly small. Newton's law of motion states

$$\vec{\mathbf{F}} = m\frac{d^2\vec{\mathbf{r}}}{dt^2}.$$

If $m \to 0$ while acceleration is left constant, then the total force on the object goes to zero. The condition $\vec{\mathbf{F}} = 0$ is more clearly understood if we consider a massless object with two forces on it, one at each end. We have

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = m\frac{d^2\vec{\mathbf{r}}}{dt^2}.$$

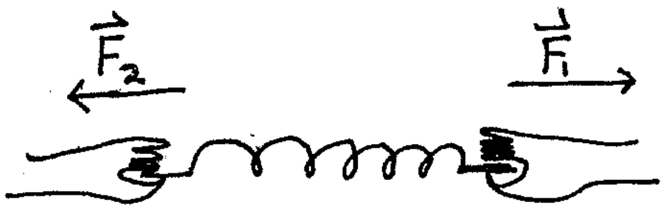
In the circumstance that m becomes negligibly small, this equation has as its unique solution

$$\vec{\mathbf{F}}_1 = -\vec{\mathbf{F}}_2.$$

That is, if a massless object is acted upon by two forces, the two forces must be equal in magnitude and point in opposite directions. Note that the two forces in question *cannot possibly* be an action-reaction pair, because they both act on the same object, while in contrast the two forces of an action-reaction pair always act upon two different objects.

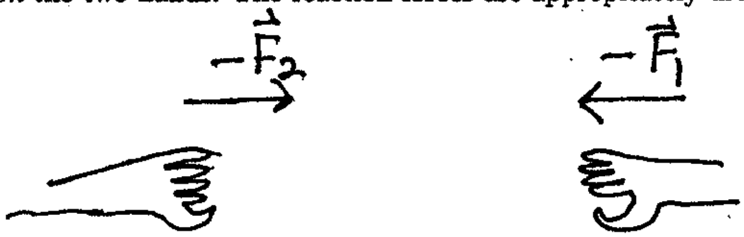
Forces on and by springs

For the most part, this course will only consider massless springs. A massless spring with two forces on it may be drawn



In this figure, the two forces that I have sketched are applied *on* the spring *by* the two hands. Because the spring in this figure is massless, the two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are required by Newton's Laws of motion to be equal in magnitude and to point in opposite directions.

The two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are indeed forces, so each of them has a reaction force. The original two forces are applied *on* the spring *by* the two hands, so the reaction forces are necessarily applied *by* the spring *on* the two hands. The reaction forces are appropriately drawn



Note that in this figure I do not show the spring, so you can see the forces more clearly, but there is indeed a spring that you cannot see supplying the two forces.

How large are the forces and reaction forces? To answer that question, we must consider how springs work.

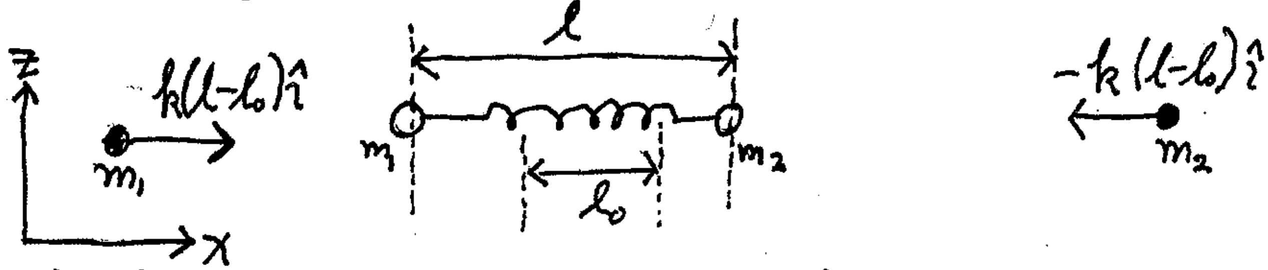
Hooke's Law Springs

The springs that we consider in this course are quite idealized relative to some springs that you encounter in nature. However, many springs are actually reasonably close to the ideal behavior we are treating here. It is generally the case that if you stretch or compress a spring by a tiny, tiny, amount it will show the ideal behavior that we are about to discuss. The idealized spring we shall now discuss is known as the Hooke's Law spring or Hookean spring.

The ideal Hookean Spring is characterized by two parameters, namely a spring constant k and an equilibrium length l_o. When we insert a spring into a mechanics problem, the spring also has an axis along which it lies, and a current length l. The spring is attached at each end to another part of the system. A massless spring with a free end that is not attached to anything else has a force $\vec{0}$ applied to its free end, and therefore also has a force $-\vec{0}$ on its other end. Such a spring has no mechanical effect on the rest of the system.

The force generated by a spring is a restoring force, which attempts to move the two attachment points until they are separated by a distance l_o. If the spring is stretched, the spring exerts a force at each end, pulling the objects to which it is attached toward its center. If the spring is compressed, the spring exerts a force on each end, pushing the objects to which it is attached away from its center.

The magnitude of the force generates by a spring is $k(l - l_o)$. The spring generates a force of this magnitude at each end of the spring, so if we take a spring of equilibrium length l_o attached to masses m_1 and m_2.



and stretch it to length $l > l_o$, the spring applies a force $+k(l-l_o)\hat{\imath}$ on mass m_1 and a force $-k(l-l_o)\hat{\imath}$ on mass m_2.

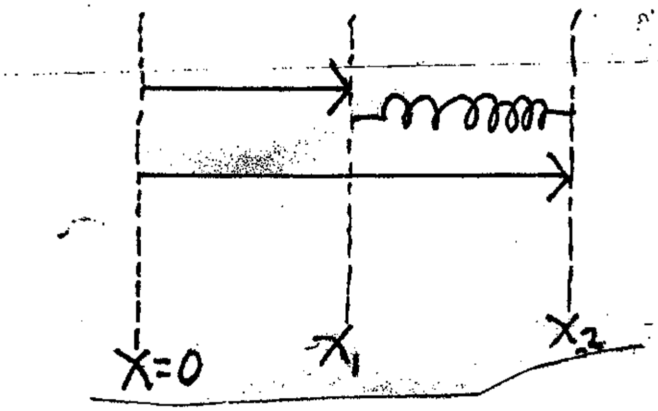
A common error is to assume that the force is divided somehow between the two masses. No, the force of magnitude $k(l - l_o)$ is applied to each mass.

Another common error is to assume that the force only appears at the end whose mass has been moved. For example, suppose we have a wall, a spring attached at one end to the wall and the other end to a mass, and a mass. Suppose we we pull the mass away from the wall. The spring pulls the mass back toward the wall. The spring also tries to pull the wall toward the mass, with a force of the same magnitude.

No matter which end of the spring is moved, the force is determined by $l - l_o$. Moving the left mass to the left, or the right mass to the right, stretches the spring, and creates a net force pulling each mass toward the center of the spring. Moving the right mass to the left, or the left mass to the right, compresses the spring, and creates a net force pushing each mass away from the center of the spring.

The examples here have put the spring along the x-axis. In general, a spring lies along some arbitrary axis in 3-dimensional space. The forces generated by the spring lie along that axis, and must be resolved into their Cartesian coordinates before we can solve any problems.

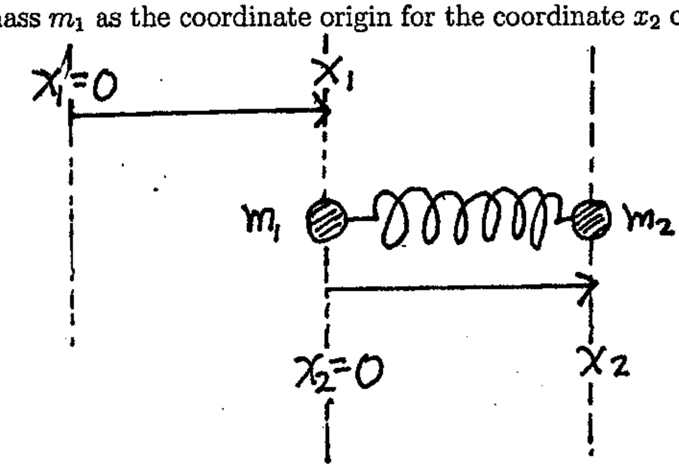
Specifying the length of the spring in terms of coordinates requires some attention to detail. For example, suppose the spring above lies along the x- axis, with the two masses at coordinates x_1 and x_2 respectively. In this case, the forces on the two masses are proportional to $k(x_2 - x_1 - l_o)$, namely



Extremely careful attention is required to ensure that x_2 and x_1 are inserted into the force equation in the correct order. If the force on the masses had been written $\pm k(x_1 - x_2 - l_o)$, the force would – incorrectly – fail to vanish when the two masses are separated by their equilibrium distance. If the masses had been labelled oppositely, interchanging m_1 and m_2, the forces would not be the same. After the interchange, the forces would correctly be written $\pm k(x_1 - x_2 - l_o)$, and forces written $\pm k(x_2 - x_1 - l_o)$ would fail to vanish when the spring had its equilibrium length.

Equal attention is needed on the signs in front of the spring constant k. When the chain is stretched the force on m_1 must be positive, because it is in the $+\hat{\imath}$ direction, while at the same time the force on m_2 must be negative, because it is in the $-\hat{\imath}$ direction. If one were eccentrically to reverse the $+x$ direction – actually, this readily occurs in problems with wedges and ramps – every single sign must be checked again, to ensure that (i) a force of the expected direction appears when each mass is moved in each direction, and (ii) the force vanishes when the spring is at its equilibrium length.

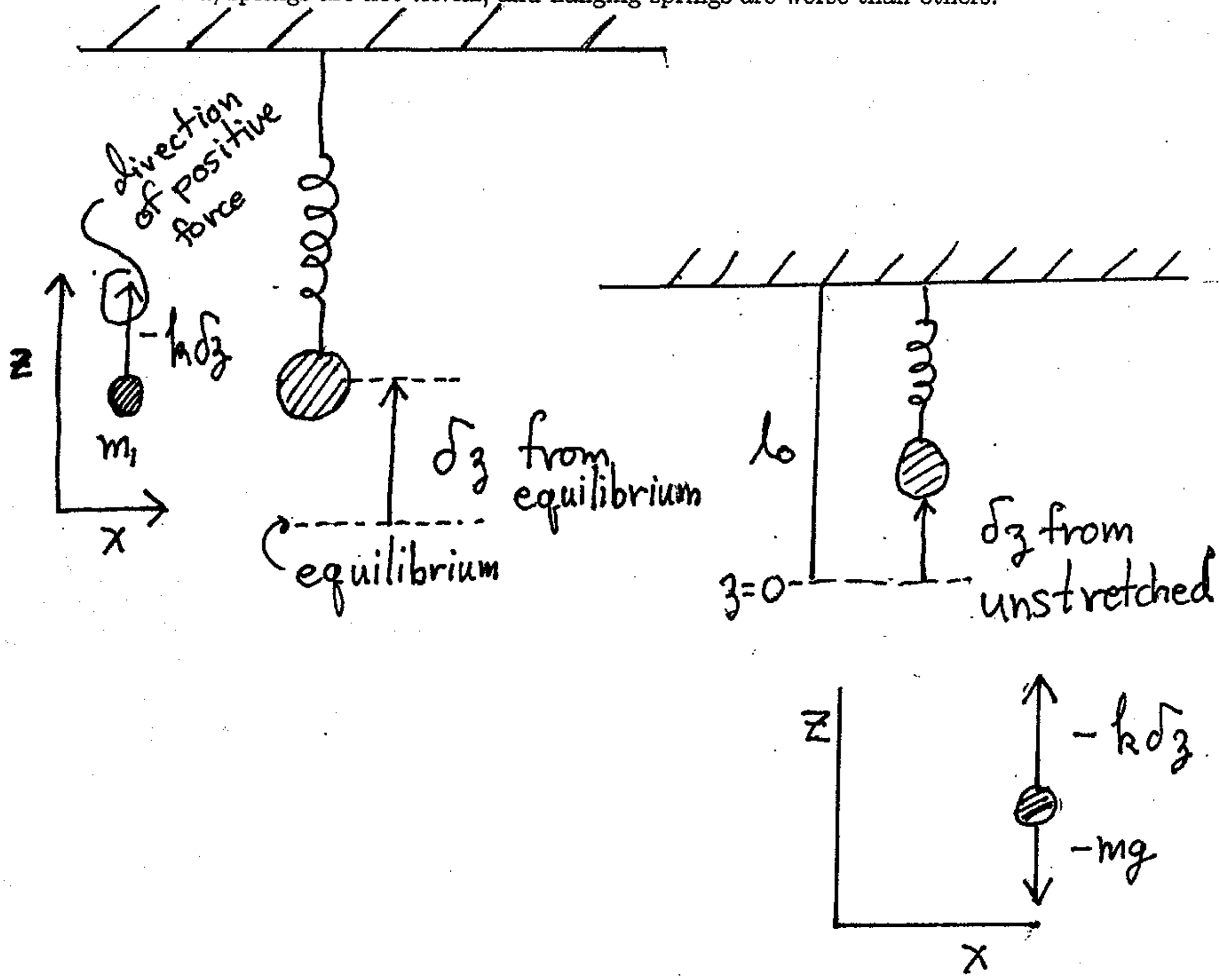
An interesting physics error appears if one defines the first mass to be at x_1, and then uses the location of mass m_1 as the coordinate origin for the coordinate x_2 of the second mass, as seen here.



It appears as though you could write the force $k(x_2 - l_o)$ with an appropriate sign. In fact, you can write the force this way, but you can't do anything with that force. The mass m_1 is free to accelerate. The coordinate x_2 is therefore the coordinate of m_2 in a non-inertial (accelerating) frame. Newton's Laws of motion are only correct in inertial reference frames. In a non-inertial frame such as the one that includes x_2, Newton's law of motion $\vec{\mathbf{F}} = md^2\vec{\mathbf{r}}/dt^2$ is incorrect. You can calculate the force in this frame, but that force and Newton's laws do not predict calculate the acceleration of m_2. The force $k(x_2 - l_o)$ is therefore useless for problem-solving.

A further complication arises for hanging masses, such as the one shown below. A hanging mass stretches the spring, so that when the mass is stationary there is a force of gravity down and a spring force up. If the mass remains stationary, the spring has been stretched to a new 'equilibrium' or 'rest' length that is greater than ℓ_o. If the mass moves away from the new "equilibrium" position there is a restoring force due to the movement of the mass away from "equilibrium". In writing $\vec{\mathbf{F}} = md^2\vec{\mathbf{r}}/dt^2$ for this problem, if you start with the spring at length l_o you have force diagram A. However, if you start with the mass in its "equilibrium" position, the length of the spring is not l_o any more. At equilibrium, the force due to the spring exactly cancels gravity. If we call δz the displacement from equilibrium, and write the force $\sim k\delta z$, the complete force diagram has the form seen in the left margin. Note that gravity does not appear in this force diagram, even though gravity still acts on the mass, because gravity has been cancelled by the force created by the spring when the spring is in its equilibrium, greater-than-l_o, length.

As I said, springs are not trivial, and hanging springs are worse than others.



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