The Hypercubical Dance:
a solution to Abbott's problem in *Flatland* ?

P.K.Aravind
Physics Department
Worcester Polytechnic Institute
Worcester, MA 01609
(email: paravind@wpi.edu)

Abbott's *Flatland* (1884) [1] is the story of the encounter between a two-dimensional creature, a Square, and a visitor from space, a Sphere. The Sphere tries to convince the Square of the reality of the third dimension, but with little success at first. Finally, after several long conversations and a few other worldly demonstrations, the Sphere succeeds in making the Square see the light (or the height?), with rather unfortunate consequences for the poor Square. *Flatland* became a big hit in the years and decades after it came out, and still continues to enjoy a robust reputation today. Much of the fallout generated by *Flatland* in the more than hundred years since it was written is documented in the annotated version of the book brought out by Stewart [2].

The mathematical (and psychological) problem tackled by Abbott in *Flatland* is that of conveying to an inhabitant of a certain space a feeling for dimensions that go beyond his or her own. This problem is very much with us today, perhaps even more so than in Abbott's time, because advances in twentieth century mathematics and physics have forced a great many exotic spaces upon our attention. To give just one example, physicists working on "string theory" [3] tell us that we may live in an eleven-dimensional universe in which seven of the dimensions are curled up into tiny loops too small for us to see. Rather than tackle this complex problem, I will consider the simpler problem of conveying to a mathematically unsophisticated audience some idea of what a four-dimensional (Euclidean) world is like. I do this by presenting a ballet based on a tesseract, or four-dimensional hypercube, that tries to bring out some of the most basic features of a four-dimensional world, particularly the relationship of the fourth dimension to the three dimensions we are already familiar with. The spirit of my demonstration is very similar to Abbott's, only it is pitched at Spacelanders who are encouraged to make the leap from three dimensions to four, just as Abbott's demonstration was pitched at a Flatlander who was encouraged to make the leap from two dimensions to three. My demonstration can be realized in two ways: either as a ballet performed on a stage, or as an animation shown on a computer screen.

I should preface this account by stating that the tesseract is a hugely popular object on which an enormous literature exists. Computer animations of the tesseract, or related objects, are very common. A Google search of "tesseract" turns up more than 86,000 hits to it. However, to the best of my knowledge, there is no animation that exploits the idea of this paper. Readers who want a general introduction to the tesseract and some of the history and literature connected with it can profitably consult the books listed in Ref. [4], among the many that exist on the subject.

There are two ways in which the demonstration of this paper differs from Abbott's, one technical and the other stylistic. The technical difference is that whereas Abbott employs the method of sections to convey the idea of a sphere to a Flatlander, I employ the method of projections to convey the idea of a hypercube to a Spacelander. In the method of sections, one shows several different "slices" of an object (i.e. cuts of it by lower dimensional hyperplanes) in an attempt to convey a feeling for the object to an inhabitant of a lower space. In the method of projections, by contrast, one casts a shadow of the entire object onto a lower dimensional space, causing its higher dimensional features to coalesce in complex ways in the space of the viewer. The stylistic difference is that whereas Abbott mainly employs skillful dialog, aided by only a few hand drawn sketches, to get his point across, I supply the viewer with a single dynamically unfolding image (or ballet) and allow his/her eye and brain to supply the necessary commentary.

The demonstration of this paper works just as well for an n-dimensional hypercube (or n-cube) as a tesseract, and so I present it first for this more general case before subsequently specializing the discussion to a tesseract. An n-cube is just the generalization of the ordinary 3-cube to higher dimensions. Exactly as a cube can be generated from a square by moving the latter parallel to itself along the third dimension, so a 4-cube (or tesseract) can be obtained from a 3-cube by moving the latter parallel to itself along the fourth dimension. An iteration of this construction up to the n-th dimension leads to the n-cube. This construction shows that an n-cube can be regarded as pair of (n-1)-cubes displaced relative to each other along the n-th dimension, with the corresponding vertices of the two (n-1)-cubes being connected by a set of parallel edges along the n-th dimension. It is easy to see that the n-cube has twice as many vertices as the (n-1)-cube, and therefore 2^n vertices in all. With a suitable Cartesian coordinate system the vertices of the n-cube can be taken as $(\pm 1, \pm 1, ..., \pm 1)$, where each of the n entries in the foregoing is either a +1 or a -1.

My animation, or "dance", begins with 2^n dancers taking up positions at the vertices of a projected n-cube (with the positions of all the vertices in the plane assumed to be distinct). The dance then proceeds via a sequence of steps, each of which is of one of the following two kinds: (1) the dancers come together in pairs along a set of parallel edges of the hypercube to kiss at the midpoints of the edges and then return to their starting points, or (2) one or more subsets of dancers move around closed loops of edges on the hypercube, with each dancer advancing from one vertex to the next on a particular loop. After the right number of steps of these two kinds, executed in the proper order, each dancer has kissed every other one and returned to his/her starting position, and the dance comes to an end.

A more complete description of the dance can be given by beginning from the fact, noted earlier, that an n-cube may be viewed as a pair of (n-1)-cubes separated from each other along the n-th dimension. The dance on an n-cube can be built up from a dance on an (n-1)-cube (assumed known) as follows:
(a) First, the dancers in each of the displaced (n-1)-cubes perform the dance among themselves.
(b) Next, the dancers in one of the displaced (n-1)-cubes trace out a Hamiltonian circuit on that cube, pausing to perform step (c) when each dancer has advanced from one vertex to the next on the circuit. (A Hamiltonian circuit on a hypercube is a path that begins from one vertex and proceeds along the hypercube edges to every other vertex only once before returning to its starting point. A simple algorithm for constructing such a circuit on an n-cube is given in Ref. [5]).
(c) During the pauses in (b), the dancers in the two displaced (n-1)-cubes approach each other along the set of parallel edges joining them to kiss at the midpoints of those edges and then return to their starting points.

The above prescription shows that a dance on an n-cube can be built up recursively from dances on lower dimensional cubes. But a dance on a 2-cube (or a "square dance") is easily devised, thereby allowing dances on all the higher dimensional cubes to be generated through repeated applications of the above three-part procedure. A dance on a 2-cube itself results from the above procedure if one defines a Hamiltonian circuit on a 1-cube (i.e. a pair of points joined by a line) as a to and fro motion from one vertex to the other and back (it should be stressed that a Hamiltonian circuit on a graph with a vertex of valence 1 is strictly impossible, and that the present definition is therefore nonstandard). The 2-cube dance obtained in this way consists of five steps, two of which are halves of a Hamiltonian circuit on a 1-cube. A Hamiltonian circuit on a 1–cube requires the dancers to pass "through" each other in the process of exchanging positions. In my computer animation of the dance, a collision between the dancers is avoided during this maneuver by having the dancers sidestep each other.

I now specialize the above scheme to n = 4 and give a detailed blueprint for a dance on a tesseract. The dance begins with $2^4 = 16$ dancers taking up positions at the vertices of the projected tesseract in Fig.1 (where the vertices have been numbered 1 through 16). Let [i,j] (a "kiss") denote a movement in which the dancers at vertices i and j approach each other along the edge joining them, kiss at the midpoint of that edge, and then return to their starting points. Also let (a,b,c,…,g,h) (a "shuffle") denote a movement in which the dancer at vertex a goes to vertex b, the one at b goes to c, …, and the one at h to a, thereby completing the cycle. The complete score of the tesseract dance then consists of the following 29 steps of kisses and shuffles (with all kisses/shuffles listed in a single step to be performed synchronously):

Step 1:	[1,2], [3,4], [5,6], [7,8], [9,10], [11,12], [13,14], [15,16]
Step 2:	[1,3], [2,4], [5,7], [6,8], [9,11], [10,12], [13,15], [14,16]
Step 3:	(3,4), (7,8), (11,12), (15,16)
Step 4:	Repeat Step 2
Step 5:	Repeat Step 3
Step 6:	[1,5], [2,6], [3,7], [4,8], [9,13], [10,14], [11,15], [12,16]
Step 7:	(5,6,8,7), (13,14,16,15)
Steps 8-13:	Repeat Steps 6 and 7 three times
Step 14:	[1,9], [2,10], [3,11], [4,12], [5,13], [6,14], [7,15], [8,16]
Step 15:	(9,10,14,13,15,16,12,11)
Steps 16-29:	Repeat Steps 14 and 15 seven times

The dance on an n-cube performed according to this scheme requires a total of $2^{n+1} - 3$ steps which, for n = 4,5 or 6, leads to 29, 61 or 125 steps, respectively. For a 5- or a 6-cube, the planar skeleton on which the dance is performed must be chosen with care to ensure that the vertices do not coincide (the simple orthographic projection used to generate Fig.1 no longer suffices for n-cubes with n > 4). The technical challenges involved in staging such a dance, and also the demands on the audience, are more severe and so this possibility will not be discussed further here.

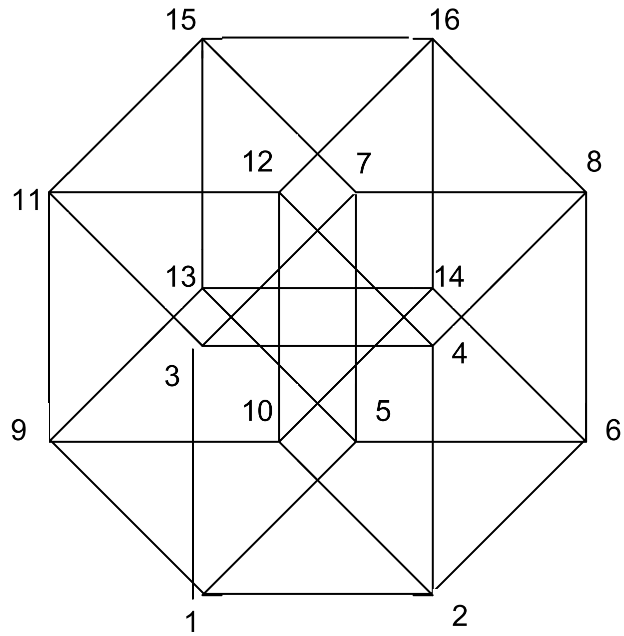

Fig.1. A hypercube with its vertices numbered 1 to 16.

The dance on the 4-cube presented above may not be totally satisfying to some viewers because the shuffles (or "Hamiltonian cycles") are distributed unevenly among the dancers. This defect is remedied in the score below in which the purpose of the last step, which consists simply of 8 exchanges of position (with a sidestep to avoid collisions), is to get the dancers back to their starting positions at the beginning of the dance.

Step 1:	[1,2], [3,4], [5,6], [7,8], [9,10], [11,12], [13,14], [15,16]
Step 2:	[1,3], [2,4], [5,7], [6,8], [9,11], [10,12], [13,15], [14,16]
Step 3:	(3,4), (7,8), (11,12), (15,16)
Step 4:	Repeat Step 2
Step 5:	(1,2), (5,6), (9,10), (13,14)
Step 6:	[1,5], [2,6], [3,7], [4,8], [9,13], [10,14], [11,15], [12,16]
Step 7:	(5,6,8,7), (13,14,16,15)
Step 8:	Repeat Step 6
Step 9:	(1,3,4,2), (9,11,12,10)
Steps 10-13:	Repeat Steps 6,7,6,9
Step 14:	[1,9], [2,10], [3,11], [4,12], [5,13], [6,14], [7,15], [8,16]
Step 15:	(9,10,14,13,15,16,12,11)
Step 16:	Repeat Step 14
Step 17:	(1,3,4,8,7,5,6,2)
Steps 18-29:	Repeat Steps 14,15,14,17,14,15,14,17,14,15,14,17
Step 30:	(1,5),(2,6),(3,7),(4,8),(9,13),(10,14),(11,15),(12,16)

REFERENCES
[1] E.A.Abbott, *Flatland: A Romance of Many Dimensions* (HarperCollins Publishers, 1994).
[2] *The Annotated Flatland* by E.A.Abbott, Introduction and notes by Ian Stewart (Perseus Publishing, Cambridge, Mass 2002).
[3] B.Greene, *The Elegant Universe: Superstrings, Hidden Dimensions and the Quest for the Ultimate Theory* (W.W.Norton and Company, 2003).
[4] C.A.Pickover, *Surfing through Hyperspace* (Oxford University Press, New York, 1999).
T.Robbin, *Fourfield: Computers, Art and the 4th Dimension* (Bullfinch Press, Boston, 1992).
T.Banchoff, *Beyond the Third Dimension* (W.H.Freeman, New York, 1990).
L.D.Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* (Princeton University Press, Princeton, 1983).
[5] A.Beck, M.N.Bleicher and D.W.Crowe, *Excursions into Mathematics* (A.K.Peters, Natick, MA, 2000).Ch.1, Sec.5.

Endnote An early version of the computer animation described in this paper was presented at the annual meeting of the Society of Literature and Science in Buffalo, New York, in October 2001. I would like to thank Professor Lance Schachterle, Associate Provost at WPI, for his interest in this work.