How Spherical Are the Archimedean Solids and Their Duals?

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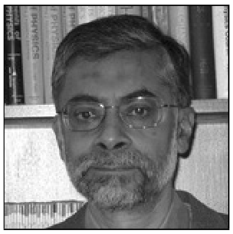

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A molecule of C-60, or a "Buckyball", consists of 60 carbon atoms arranged at the vertices of a truncated icosahedron (see Figure 1), with the atoms being held in place by covalent bonds. The Buckyball has a very spherical appearance and its sphericity is the basis of several of its applications, such as a nanoscale ball bearing or a molecular cage to trap other atoms [10]. The buckyball pattern also occurs on the surface of many soccer balls. The huge interest in the Buckyball leads one to ask whether the truncated icosahedron is the most spherical of the elementary geometrical forms or whether there are others that are superior to it. That is the question I address in this paper. The reader who is familiar with the Archimedean solids and their duals is invited to try and guess the most spherical among them before reading on.

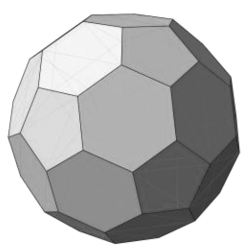

Figure 1. The truncated icosahedron

Measuring sphericity

A measure of the sphericity of a convex solid was introduced by G. Polya [**7**], who termed it the *isoperimetric quotient*, or IQ. Polya defined the IQ of a convex solid as $\mathrm{IQ} = 36\pi V^2/S^3$, where V and S are the volume and surface area of the solid. This makes the IQ of a sphere 1, and that of any other convex solid less than 1, by the

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isoperimetric theorem (see [**7**, Ch. X] and Problem 43 at the end of this chapter). The more closely the IQ of a solid approaches 1, the more spherical it is.

The truncated icosahedron is only one of thirteen semiregular or "Archimedean" polyhedra [**13**] (excluding the prisms and antiprisms, which we will ignore). The key feature distinguishing an Archimedean solid from a regular or "Platonic" solid is that it has two or more types of regular polygons for its faces. A systematic derivation of all the Archimedean polyhedra is given in [**6**] and [**12**], based on Euler's formula for polyhedra and the requirement that the same arrangement of polygonal faces occur at each vertex. The derivation shows that the semiregular polyhedra have either two or three different types of regular polygons for their faces. We can therefore describe an Archimedean polyhedron by its Schlafli symbol $\{p_1, q_1;\ p_2, q_2\}$ or $\{p_1, q_1;\ p_2, q_2;\ p_3, q_3\}$, which expresses the fact that q_1 polygons of p_1 sides and q_2 polygons of p_2 sides (and perhaps q_3 polygons of p_3 sides) meet at each vertex. This symbol is slightly ambiguous because it does not specify the order in which the polygons occur around a vertex. However it is perfectly adequate for our purposes because it encodes the essential information in terms of which our results are presented.

Table 1 shows the thirteen Archimedean polyhedra, together with their Schlafli symbols. Straightforward counting arguments [**6, 12**] show that the number of vertices (v),

Table 1. The thirteen Archimedean polyhedra and their duals.

Polyhedron ⇓	Schlafli Symbol	v	f	e	
Truncated Tetrahedron	{3, 1; 6, 2}	12	8	18	Triakis Tetrahedron
Truncated Octahedron	4,1;6,2	24	14	36	Tetrakis Hexahedron
Truncated Cube	3,1;8,2	24	14	36	Triakis Octahedron
Truncated Icosahedron	5,1;6,2	60	32	90	Pentakis Dodecahedron
Truncated Dodecahedron	3,1;10,2	60	32	90	Triakis Icosahedron
Cuboctahedron	{3, 2; 4, 2}	12	14	24	Rhombic Dodecahedron
Icosidodecahedron	{3, 2; 5, 2}	30	32	60	Rhombic Triacontahedron
Snub Cube	{3, 4; 4, 1}	24	38	60	Pentagonal Icositetrahedron
Snub Dodecahedron	{3, 4; 5, 1}	60	92	150	Pentagonal Hexecontahedron
Small Rhombicuboctahedron	{3, 1; 4, 3}	24	26	48	Trapezoidal Icositetrahedron
Small Rhombicosidodecahedron	{3, 1; 4, 2; 5, 1}	60	62	120	Trapezoidal Hexecontahedron
Great Rhombicuboctahedron	{4, 1; 6, 1; 8, 1}	48	26	72	Hexakis Octahedron
Great Rhombicosidodecahedron	{4, 1; 6, 1; 10, 1}	120	62	180	Hexakis Icosahedron
		f	v	e	⇑ Dual Polyhedron

edges (e), and faces (f_n) of any polyhedron are given in terms of the integers in its Schlafli symbol by

$$v = \frac{4}{2 - \sum_n (1 - 2/p_n) q_n}, \quad e = \frac{1}{2} v \sum_n q_n, \quad \text{and} \quad f_n = \frac{v q_n}{p_n}, \tag{1}$$

where the sums in the first two expressions are over all the different types of polygons in the polyhedron and therefore consist of either two or three terms (i.e., $n = 1, 2$ or $n = 1, 2, 3$). The symbol f_n denotes the number of polygonal faces with p_n sides, with the total number of faces, f, being given by $f_1 + f_2$ or $f_1 + f_2 + f_3$. Table 1 lists the values of v, f, and e for each of the Archimedean polyhedra. For example, the truncated icosahedron has $v = 60$, $f = 32$ (with $f_1 = 12$ and $f_2 = 20$), and $e = 90$, and Euler's formula, $v + f - e = 2$, is satisfied because $60 + 32 - 90 = 2$.

Every Archimedean polyhedron has a dual that can be obtained from it by polar reciprocation. The number of vertices (or faces) of the dual is equal to the number of faces (or vertices) of the original polyhedron, while the number of edges of the two are the same. The last column of Table 1 shows the duals of all the polyhedra in the first column. The faces of any dual are always identical (but never regular) polygons. The duals of the Archimedean solids are sometimes called the Catalan solids.

In this paper I develop closed form expressions for the surface area and volume of any Archimedean solid (and its dual) in terms of the integers in its Schlafli symbol, and use these to compute the IQs of the 26 polyhedra in Table 1. The areas and volumes could have been obtained in other ways—by making use of the metrical properties of these polyhedra tabulated in [**2, 3, 8, 14, 15**] or, even more simply, by asking *Maple* or *Mathematica* to return the results. However the technique I present here dispenses with the need for either a tedious case by case calculation or reliance upon a "black box" that spits out the answer with no indication of how it was obtained. Technique aside, my main reason for undertaking this calculation was to find the answer to the riddle, "Buckyball, oh Buckyball, are you the roundest of them all?" The answer surprised me. Perhaps it will surprise the reader also.

Surface area and volume of an Archimedean polyhedron

The surface area of an Archimedean polyhedron is easily calculated by summing the areas of all its polygonal faces. The result is

$$S = l^2 \sum_n p_n f_n \cot(\pi/p_n), \tag{2}$$

where l is half the edge length of the polyhedron and the sum consists of either two or three terms. Equations (1) and (2) allow the surface area to be calculated in terms of the integers in the Schlafli symbol (and the edge length of the polyhedron). I turn next to the task of calculating the volume.

The vertices of any Archimedean polyhedron lie on a sphere (the *circumsphere*) and the midpoints of its edges on another concentric sphere (the *midsphere*) whose common center is the center of symmetry of the polyhedron. I denote the radius of the circumsphere (the *circumradius*) by R_0 and that of the midsphere (the *midradius*) by R_1. (Unlike a regular polyhedron, a semiregular one does *not* have a third concentric sphere, the *insphere*, that touches all its faces). To calculate the volume, we dissect the polyhedron into pyramids based on its faces, whose apexes meet at its center, and

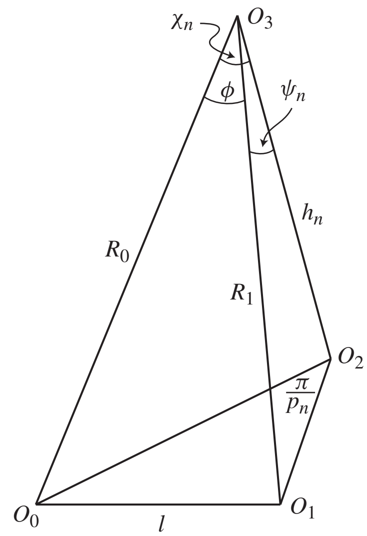


Figure 2. The quadrirectangular tetrahedron $O_0O_1O_2O_3$. Angles $\angle O_0O_1O_2$, $\angle O_0O_2O_3$, $\angle O_1O_2O_3$, and $\angle O_0O_1O_3$ are 90°.

then further dissect each pyramid into tetrahedra by means of cuts along all its symmetry planes. The elementary tetrahedra obtained in this way have only right triangles for faces and are termed *quadrirectangular* tetrahedra in [**2**]. The quadrirectangular tetrahedra yielded by any pyramid are of just two types that are enantiomorphs of each other. A typical quadrirectangular tetrahedron, obtained by dissecting a pyramid with a polygonal base of p_n sides, is shown in Figure 2; its vertices are the center of the polyhedron, O_3, the center of a polygonal face, O_2, the center of an edge, O_1, and a vertex, O_0. The important lengths in this figure are $O_0O_1 = l$, $O_0O_3 = R_0$, $O_1O_3 = R_1$, and $O_2O_3 = h_n$ (the altitude of the tetrahedron). The right angles are $\angle O_0O_1O_2$, $\angle O_0O_2O_3$, $\angle O_1O_2O_3$, and $\angle O_0O_1O_3$, and the other important angles are $\angle O_0O_2O_1 = \pi/p_n$, $\angle O_0O_3O_1 = \phi$, $\angle O_0O_3O_2 = \chi_n$, and $\angle O_1O_3O_2 = \psi_n$. The Pythagorean theorem, along with some simple trigonometry, allow us to deduce that

$$l = R_0 \sin\phi, \quad R_1 = R_0\cos\phi, \quad \text{and} \quad h_n = l\sqrt{\csc^2\phi - \csc^2(\pi/p_n)}. \tag{3}$$

The volume of the polyhedron is the sum of the volumes of its basal pyramids, with the volume of a basal pyramid being a multiple of the volume of a quadrirectangular tetrahedron:

$$V = \frac{1}{3}l^3 \sum_n p_n f_n \cot(\pi/p_n)\sqrt{\csc^2\phi - \csc^2(\pi/p_n)}. \tag{4}$$

It remains only to calculate the angle ϕ in terms of the Schlafli symbol. This can be done using some spherical trigonometry, as explained in the next paragraph. However the reader who finds this excursion distasteful can skip directly to the results in equations (7a)–(7f) below, which give closed form expressions for ϕ in terms of the Schlafli symbols for all thirteen Archimedean polyhedra.

To determine ϕ, as well as χ_n and ψ_n, we project the basal triangle $O_0O_1O_2$ of the quadrirectangular tetrahedron onto the circumsphere of the polyhedron to obtain the spherical triangle $O_0Q_1Q_2$ (with the point O_0 being unchanged but O_1 and O_2

projecting into Q_1 and Q_2, respectively). This triangle is depicted in Figure 3. The angle δ_n appearing in this triangle is half the vertex angle of the *spherical* polygon into which a polygonal face of p_n sides gets mapped by this projection. Applying spherical trigonometry to this triangle leads to the relations

$$\cos\phi = \cos(\pi/p_n)\csc\delta_n, \quad \cos\psi_n = \csc(\pi/p_n)\cos\delta_n,$$
$$\text{and} \quad \cos\chi_n = \cot(\pi/p_n)\cot\delta_n, \qquad (n = 1, 2 \text{ or } 1, 2, 3). \tag{5}$$

[Recall that if ABC is a spherical triangle with a, b, and c being the sides opposite angles A, B, and C, then $\cos a = (\cos A + \cos B\cos C)/\sin B\sin C$, with similar relations for the other two sides. Applying these formulas to the spherical triangle of Figure 3 leads immediately to the above results].

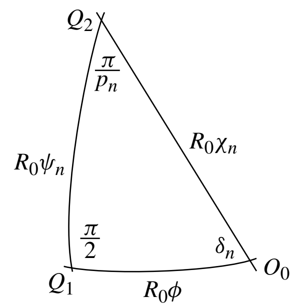


Figure 3. Spherical triangle: the projection of the basal triangle $O_0O_1O_2$ of Figure 2 onto the circumsphere of the polyhedron.

The six or nine equations in (5) must be supplemented by the relation

$$q_1\delta_1 + q_2\delta_2\,(+q_3\delta_3) = \pi, \tag{6}$$

which expresses the fact that when the polyhedron is projected on to its circumsphere, the sum of the semi-vertex angles of all the spherical polygons that meet at a vertex must be π. Equations (5) and (6) are a set of 7 (or 10) equations for the 7 (or 10) unknowns χ_n, ψ_n, δ_n, and ϕ, all of which lie between 0 and $\pi/2$ (and so can be fixed without ambiguity).

Equations (5) and (6) can be solved in closed form for all thirteen Archimedean polyhedra. It is convenient to solve for ϕ first, and then obtain the other angles from it. In solving for ϕ, I found it convenient to group the polyhedra having the same values of q_1, q_2 (and possibly q_3) together, since the expressions for ϕ then have the same general form. The six different cases are:

I. $q_1 = 1, q_2 = 2$: $$\phi = \cos^{-1}\left[\frac{2c_2^2}{\sqrt{4c_2^2 - c_1^2}}\right] \tag{7a}$$

II. $q_1 = 2, q_2 = 2$: $$\phi = \cos^{-1}\left(\sqrt{c_1^2 + c_2^2}\right) \tag{7b}$$

III. $q_1 = 1, q_2 = 3$: $$\phi = \cos^{-1}\left[\sqrt{\frac{4c_2^3}{3c_2 - c_1}}\right] \tag{7c}$$

IV. $q_1 = 4, q_2 = 1$: $$\phi = \cos^{-1}\left[c_1 \Big/ \sqrt{\frac{2}{3} - \frac{\beta^{2/3}}{36} - \frac{1}{\beta^{2/3}}}\right] \tag{7d}$$

V. $q_1 = 1, q_2 = 1, q_3 = 1$: $$\phi = \cos^{-1}\left[\frac{2c_1c_2c_3}{\sqrt{\left(c_1^2 + c_2^2 + c_3^2\right)^2 - 2\left(c_1^4 + c_2^4 + c_3^4\right)}}\right] \tag{7e}$$

VI. $q_1 = 1, q_2 = 2, q_3 = 1$: $$\phi = \cos^{-1}\left(2c_2\sqrt{\frac{c_2^2 + c_1c_3}{(2c_2)^2 - (c_3 - c_1)^2}}\right), \tag{7f}$$

where $c_i = \cos(\pi/p_i)$, $i = 1, 2, 3$, and $\beta = (27c_2 + 3\sqrt{81c_2^2 - 96c_1^2})/2c_1$. The number of polyhedra covered by each of the cases is 5, 2, 1, 2, 2, and 1, respectively. Substituting the values of p_1, p_2 (and possibly p_3) into the above formulas yields expressions for ϕ for all the thirteen polyhedra, allowing their volumes to be calculated from (4).

The results for ϕ obtained above allow *all* the metrical properties of the Archimedean polyhedra to be calculated. With ϕ in hand, δ_n, ψ_n, and χ_n can be calculated from (5) and (6) and the lengths R_0,R_1, and h_n from (3). The dihedral angle between the faces p_1 and p_2 can then be found as $\pi - (\psi_1 + \psi_2)$. The numerical values of the preceding quantities found in this way agree with the values listed in [**3**], [**8**], and [**15**].

Surface areas and volumes of the Archimedean duals

The dual of any Archimedean polyhedron can be obtained by a construction due to Dorman Luke [**3, 14**]. This construction brings out the following important properties of the dual:

(a) The number of faces (or vertices) of the dual is equal to the number of vertices (or faces) of the polyhedron, while the number of edges of the two are the same.

(b) The dual has the same center and the same symmetry group as the polyhedron.

(c) The dual is facially regular but not vertex regular. That is, the faces of the dual are identical (but generally not regular) polygons, while its vertices are not all surrounded alike.

(d) The faces of the dual are all at the same distance from its center; thus there is a sphere, the *insphere*, concentric with the circum- and mid-spheres of the original polyhedron, that touches all the faces of the dual. The radius of the insphere is $R_2 = R_0\cos^2\phi$.

(e) A circle of radius $r_c = R_0 \sin\phi\cos\phi$ can be inscribed in every face of the dual.

(f) The two or three distinct face angles of the dual are given by $\pi - 2\delta_n$, there being q_n angles of each type (with $n = 1, 2$ or $n = 1, 2, 3$).

Properties (e) and (f) allow us to calculate the area of a face of the dual by dissecting it into quadrilaterals obtained by dropping perpendiculars from the center of its inscribed circle on to its sides. Each quadrilateral thus obtained is bounded by two radii and two tangents of the inscribed circle. This is illustrated in Figure 4 for a dual with the rhombic face *XYZW*, which is dissected into four quadrilaterals by this construction. One of the quadrilaterals, *OAXB*, has $OA = OB = r_c$, $\angle AXB = \pi - 2\delta_n$, $\angle AOB = 2\delta_n$, and $\angle OAX = \angle OBX = 90°$. Its area is thus $r_c^2\tan\delta_n$, and because there are q_n quadrilaterals like this in a face, the area of the face is given by $\sum_n q_n {r_c}^2\tan\delta_n$. Multiplying this expression by the number of faces of the dual, f_d, and eliminating r_c

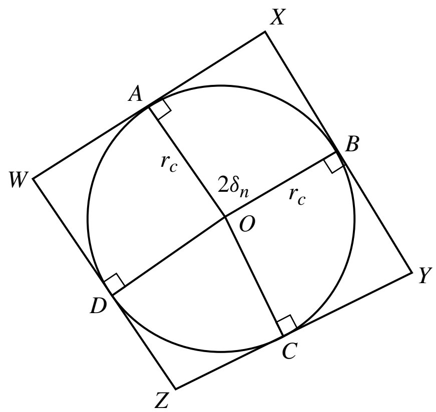


Figure 4. A rhombic face *XYZW* of the dual polyhedron, with its inscribed circle centered at O.

in favor of R_2 gives the total surface area of the dual as

$$S_d = f_d R_2^2 \tan^2 \phi \sum_n q_n \tan \delta_n, \qquad n = 1, 2 \text{ or } n = 1, 2, 3. \tag{8}$$

The volume of the dual can be calculated by decomposing it into pyramids based on its faces whose apexes meet at its center. Since all the pyramids have the same altitudeR_2, the volume of the dual is simply

$$V_d = \frac{1}{3} R_2 S_d. \tag{9}$$

Equations (8) and (9) allow the surface area and volume of the dual to be calculated in terms of the Schlafli symbol of the original polyhedron if one keeps in mind that f_d is given by (1a) and that $R_2 = R_0 \cos^2 \phi$, where R_0 and ϕ have already been determined for the original polyhedron.

Present approach compared to *Mathematica* and *Maple*

The formulas of the last two sections allow the surface area and volume of any Archimedean polyhedron or its dual to be expressed in closed form in terms of the quantities $c_1 = \cos(\pi/p_1)$, $c_2 = \cos(\pi/p_2)$ (and possibly $c_3 = \cos(\pi/p_3)$). If the cosines are expressed as radicals, then the areas and volumes can also be expressed in terms of radicals. Consider, for example, the truncated icosahedron, with $p_1 = 5$ and $p_2 = 6$ and the edge length taken to be 2. The formulas of the last two sections then yield its surface area and volume as $S = 12\big[10\sqrt{3} + \sqrt{5}\sqrt{5 + 2\sqrt{5}}\,\big]$ and $V = 250 + 86\sqrt{5}$, respectively, in agreement with *Maple* and *Mathematica*. In some cases a fair amount of simplification must be done to reconcile our expressions for the area and volume with those yielded by these programs. An interesting divergence between the present approach and the programs occurs for the snub dodecahedron and its dual, the pentagonal hexecontahedron. *Mathematica* gives the volumes of the two solids, as well as the surface area of the latter, as the roots of 12th order polynomial equations with large coefficients. My approach gives compact closed form expressions in terms of c_1 and c_2, although attempting to recast them in terms of radicals only leads to very long and messy expressions. Nevertheless, the numerical values yielded by the two approaches are identical.

IQs of the Archimedean solids and their duals

Using the results obtained above, I calculated the IQs of the Archimedean solids and their duals and have listed the results in Table 2. The correctness of these results can be checked in a few simple limiting cases, which I would like to mention first.

Table 2. Isoperimetric quotients of the thirteen Archimedean polyhedra and their duals.

Polyhedron	IQ	Dual Polyhedron	IQ
Truncated Tetrahedron	0.466229	Triakis Tetrahedron	0.645836
Truncated Octahedron	0.753367	Tetrakis Hexahedron	0.842978
Truncated Cube	0.613028	Triakis Octahedron	0.790028
Truncated Icosahedron	0.903171	Pentakis Dodecahedron	0.939707
Truncated Dodecahedron	0.794055	Triakis Icosahedron	0.905181
Cuboctahedron	0.741211	Rhombic Dodecahedron	0.740480
Icosidodecahedron	0.860151	Rhombic Triacontahedron	0.887200
Snub Cube	0.899181	Pentagonal Icositetrahedron	0.872628
Snub Dodecahedron	0.946999	Pentagonal Hexecontahedron	0.945897
Small Rhombicuboctahedron	0.868468	Trapezoidal Icositetrahedron	0.869774
Great Rhombicuboctahedron	0.839004	Hexakis Octahedron	0.910066
Small Rhombicosidodecahedron	0.938995	Trapezoidal Hexecontahedron	0.945852
Great Rhombicosidodecahedron	0.913556	Hexakis Icosahedron	0.957765

Consider the regular (or Platonic) solids, which have only a single type of polygonal face, and for which the Schlafli symbol collapses to the single pair of integers $\{p, q\}$. Equations (1), (5), and (6) then show that $f_p = 4q/(2p + 2q - pq)$, $\cos\phi = \cos(\pi/p)\csc(\pi/q)$, and $\delta = \pi/q$. Using these in (2) and (4) to calculate the surface area and volume and forming the expression for the IQ leads to the result

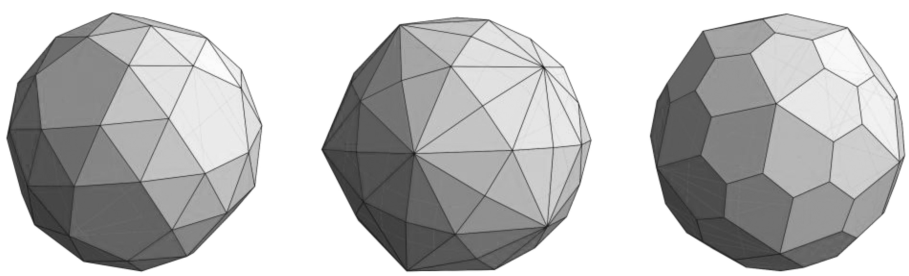
$$\text{IQ} = \frac{\pi(2p + 2q - pq)\cot(\pi/p)\cos^2(\pi/q)}{pq[\sin^2(\pi/q) - \cos^2(\pi/p)]}. \tag{10}$$

Putting $\{p, q\} = \{3, 3\}, \{3, 4\}, \{(4, 3\}, \{3, 5\}$, or $\{5, 3\}$ in (10) gives back the numerical values of the IQs of the tetrahedron, octahedron, cube, icosahedron and dodecahedron given by Polya [**7**].

Next let us calculate the IQ of a rhombic dodecahedron, which is the dual of the cuboctahedron. This can be done very simply if one uses the fact that a rhombic dodecahedron can be built up from two identical cubes by dissecting one of them into six square pyramids and gluing one pyramid on to each face of the other cube. This construction shows that the volume and surface area of the dodecahedron are 2 and

$6\sqrt{2}$, respectively, if the cubes from which it is constructed have unit edge length. It follows that the IQ is $\pi/3\sqrt{2} = 0.7405$, in agreement with Table 2. As yet another check, we calculate the IQ of the dual of the regular polyhedron $\{p, q\}$. We do this by putting $f_d = 4p/(2p + 2q - pq)$, $\phi = \cos^{-1}[\cos(\pi/p)\csc(\pi/q)]$ and $\delta = \pi/q$ into (8) and (9), noting that the summations now consist of a single term. On forming the expression for the IQ, we find that we get back (10) but with p and q interchanged; but this is just the IQ of $\{q, p\}$, the expected result.

Table 2 shows that the truncated icosahedron is *not* the most spherical of these solids, but that three other Archimedean solids and six Catalan solids have IQs larger than it. The most spherical of the Archimedeans is the snub dodecahedron (IQ = 0.946999) and the most spherical of the Catalans is the hexakis icosahedron (IQ = 0.957765). These solids are shown in Figure 5a and 5b. It is a general, but not infallible, rule that the IQ increases with the number of faces of the polyhedron. Interestingly, the pentagonal and trapezoidal hexecontahedra, both with only 60 faces, have IQs that are nearly equal and only slightly less than those of the two leaders in the list, both of which have considerably more faces. The pentagonal hexecontahedron (see Figure 5c) comes in two enantiomorphous forms (like the snub dodecahedron, of which it is the dual), and superposing these forms on one another produces a design that is quite riveting [**11**].



(a) Snub dodecahedron (b) Hexakis icosahedron (c) Pentagonal hexecontahedron

Figure 5. The most spherical semi-regular solids.

Aside from their decorative and aesthetic appeal, what are some of the applications of these solids? The Archimedean polyhedra crop up frequently in chemistry, as molecular structures of various kinds [**5**]. The truncated icosahedron has of course captured the lion's share of the attention in recent years, with the discovery of the fullerenes [**4**]. Physicists and crystallographers are interested in the truncated octahedron and the rhombic dodecahedron because they are the Wigner-Seitz cells (or Dirichlet regions) of the body-centered and face-centered cubic lattices [**1**]. In cubic close packing, each sphere lies at the center of twelve other spheres whose centers lie at the vertices of a cuboctahedron. The rhombic triacontahedron is of interest in connection with quasicrystals, which are three-dimensional generalizations of the Penrose tilings [**9**]. The results of this paper demonstrate that some of the Archimedean and Catalan solids are better approximations of the sphere than the truncated icosahedron. Whether this fact might find any application is a question that I leave as food for thought. The occurrences of the Archimedean and Catalan solids in the natural and man-made worlds are many and varied, and it is probably safe to say that there are still exciting discoveries that remain to be made.

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Summary. The Isoperimetric Quotient, or IQ, introduced by G. Polya, characterizes the degree of sphericity of a convex solid. This paper obtains closed form expressions for the surface area and volume of any Archimedean polyhedron in terms of the integers specifying the type and number of regular polygons occurring around each vertex. Similar results are obtained for the Catalan solids, which are the duals of the Archimedeans. These results are used to compute the IQs of the Archimedean and Catalan solid and it is found that nine of them have greater sphericity than the truncated icosahedron, the solid which serves as the geometric framework for a molecule of C-60, or "Buckyball", and which is naively regarded as very spherical.

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