BORROMEAN ENTANGLEMENT OF THE GHZ STATE*

In this paper, I will point out some curious connections between entangled quantum states and classical knot configurations. In particular, I will show that the entanglement of the particles in a Greenberger–Horne–Zeilinger[1] (GHZ) state is modelled by a set of interlinked rings known as the Borromean rings. It is widely acknowledged that the non-local properties of multiparticle quantum states (such as the GHZ state) derive from their entanglement. By the entanglement of a multiparticle state, I mean simply that the wave function of the state cannot be written as a product of wave functions of the individual particles. Now one of the images conjured up by the term "entanglement" is that of a tangled collection of strings. This led me to enquire whether there might be any similarities between the entanglement of quantum particles and the entanglement of loops of string, or whether the expectation of such a connection is completely far-fetched.

I will begin by looking at the three-particle GHZ state and pointing out a similarity between it and the Borromean rings, shown in Figure 1. The GHZ state of three spin $\frac{1}{2}$ particles has the form:

$$| \Psi \rangle = \frac{1}{\sqrt{2}}(|\beta_1\rangle\, |\beta_2\rangle\, |\beta_3\rangle - |\alpha_1\rangle\, |\alpha_2\rangle\, |\alpha_3\rangle) \tag{1}$$

where the subscripts 1, 2 and 3 refer to the different particles and $|\beta_i\rangle$ and $|\alpha_i\rangle$ denote the spin-up and spin-down states of particle i along the z-direction.[2] Before making the desired connection, I say a few words about the Borromean rings. As is evident from Figure 1, these are a set of three interlinked rings that cannot be pulled apart. However, if any one of the rings is cut, the other two can be separated without difficulty. The Borromean rings[3] are named after the princely Italian

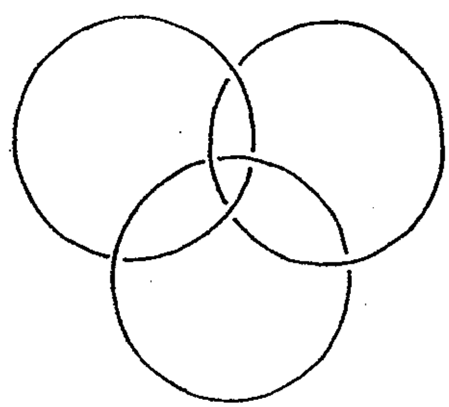

Figure 1. The Borromean rings. A break in a ring indicates that it passes under the ring that crosses it. The three rings in this figure cannot be pulled apart, but if any one is cut the other two can be separated easily.

*It is a pleasure to contribute this essay to this volume honouring Abner Shimony. It was my good fortune to become acquainted with Abner a few years back and I have benefited in many ways from my interaction with him ever since.

R.S. Cohen, et al. (eds), Potentiality, Entanglement and Passion-at-a-Distance, 53–59

family of the Borromeos and occur as heraldic symbols on the family coat-of-arms. Visitors to a castle on one of the Borromean islands (in Lake Maggiore in northern Italy) can see the rings cut into the stonework. Some have identified the rings with the indivisible Trinity whereas to others they represent the motto "United we stand, divided we fall", since if one of the rings is cut the other two fall apart.

But back to physics. We make the following associations between the GHZ state and the Borromean rings: (1) each particle is associated with a ring; (2) measuring the spin of a particle along the z-direction is regarded as the equivalent of cutting the corresponding ring; and (3) the entanglement of any set of particles is modelled by the inability to separate the corresponding rings. With these associations, one finds that the entanglement of the particles in the GHZ state is faithfully mirrored in the Borromean rings. To see this explicitly, note that as long as no measurement is made on particle 1, particles 2 and 3 are in an entangled state because their reduced density operator

$$\rho = \frac{1}{2} |\beta_2\rangle\, |\beta_3\rangle \langle \beta_2 |\, \langle \beta_3 | + \frac{1}{2} |\alpha_2\rangle\, |\alpha_3\rangle \langle \alpha_2 |\, \langle \alpha_3 | \qquad (2)$$

cannot be written as a product of density operators of particles 2 and 3; this is similar to the inability to pull two of the rings apart if the third one is left undisturbed. However, if a measurement is made on the spin of 1 along the z-direction, then regardless of the outcome, particles 2 and 3 become disentangled; this is similar to the ability to pull two of the rings apart if the third one is cut. Just as the GHZ state is symmetrical in all three particles, so are the Borromean rings symmetrical in all three rings. It should be stressed that although the state (2) is entangled (in the sense of being non-factorizable) it is nevertheless not non-local (in the sense of violating any Bell-like inequalities[4]. It is the non-factorizable property of (2) that is captured in Figure 1 through the inability to pull apart the corresponding rings.

The above correspondence is quite gratifying. However, if we try to push it any further, matters become more complicated (but also more interesting). Suppose that instead of measuring spins along the z-direction, we measure them along the x-direction. Then it is natural to associate the cutting of a knot with a spin measurement on the corresponding particle along the x-direction. With this altered meaning of the phrase "cutting a knot", one finds that the GHZ state is no longer modelled by the Borromean rings in Figure 1. To see this, note that the GHZ state can be rewritten as:

$$\begin{aligned} |\Psi\rangle &= \frac{|\beta_{1x}\rangle}{\sqrt{2}} \left(\frac{|\beta_2\rangle\, |\beta_3\rangle - |\alpha_2\rangle\, |\alpha_3\rangle}{\sqrt{2}} \right) \\ &+ \frac{|\alpha_{1x}\rangle}{\sqrt{2}} \left(\frac{|\beta_2\rangle\, |\beta_3\rangle + |\alpha_2\rangle\, |\alpha_3\rangle}{\sqrt{2}} \right) \end{aligned} \qquad (3)$$

where $|\beta_{1x}\rangle$ and $|\alpha_{1x}\rangle$ denote the spin-up and spin-down states of particle 1 along the x-direction. It is evident from (3) that if the spin of 1 is measured along x then, irrespective of the outcome, 2 and 3 are thrown into a maximally entangled state. The knot configuration that now models the GHZ state is shown in Figure 2. Note,

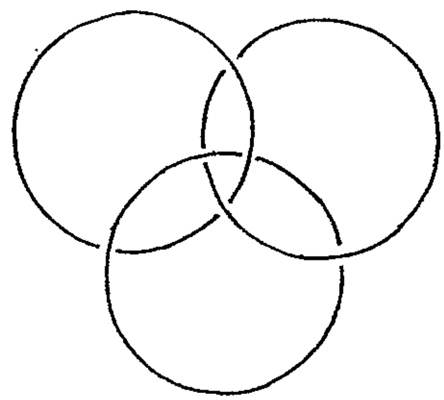

Figure 2. An interlocked set of three rings (the "three-Hopf rings"). If any ring is cut, the other two are still linked, so this configuration differs from the Borromean one of Figure. 1.

in contrast to the Borromean rings, that each pair of rings is linked and cannot be separated even if the third ring is cut. The non-factorizability of particles 2 and 3 after a spin-x measurement on 1 is modelled by the linkage of the corresponding rings in Figure 2. Again, the symmetry of the GHZ state in all three particles is mirrored in the symmetry of Figure 2 in all three rings. For brevity, I will refer to the configuration in Figure 2 as the "three-Hopf rings", since a pair of interlinked rings are usually termed Hopf rings.

What if we were to measure spins along some arbitrary direction instead of the z- or x-directions? Then it is easy to show that the GHZ state continues to be modelled by the three-Hopf rings of Figure 2 but with the difference that the state of any two particles after a measurement on the third is no longer a maximally entangled state. As it stands, Figure 2 has no means of conveying the strength of the entanglement between a pair of particles after a measurement on the remaining particle is made.

The above discussion demonstrates that an entangled quantum state generally corresponds to more than one knot configuration. In establishing the connection between quantum states and knots at all, it is first necessary to give a quantum mechanical meaning to the mathematical act of cutting a knot. A quantum process that suggests itself naturally in this regard is the act of measuring the spin (or some other observable) of a particle, followed by the collapse of the multiparticle wave function. However, the observable to be measured can be chosen in many different ways (for example, the spin can be measured along any direction in space) and thus there seems to be no unique quantum process that corresponds to the mathematical act of cutting a knot. If we represent quantum particles by knots and entangled quantum states by interlinked knots, then it would seem that there are as many ways of cutting each quantum knot as there are of making measurements on that particle. Thus, depending on the observables that one measures for each particle, one will obtain different knot configurations that model the same quantum state. The GHZ state corresponds to either the Borromean or three-Hopf rings, depending on whether one measures the spins of the particles along the z-direction or some other direction in space.

Let us turn next to a more peculiar example. Consider the state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|\beta_1\rangle\{|\beta_2\rangle|\alpha_3\rangle + |\alpha_2\rangle|\beta_3\rangle\} + |\alpha_1\rangle\{|\beta_2\rangle|\beta_3\rangle\}) \tag{4}$$

with the same notational conventions as in (1). The state (4) is a three-particle version of a four-particle state introduced by Zeilinger, Horne and Greenberger[5] to show that entanglement itself is an entangled quantity. Suppose that, in the state (4), we measure the spin of particle 1 along the z-direction. Then, as is evident from (4), if particle 1 is found to have spin up, particles 2 and 3 are left in a maximally entangled state, but if 1 is found to have spin down, 2 and 3 get completely disentangled. As Zeilinger, Horne and Greenberger emphasize, particle 1 can be in a space-time region remote from particles 2 and 3, yet a chance event befalling 1 strongly influences the mutual relationship of 2 and 3. This is a very peculiar situation indeed! Is there a knot configuration that models the state (4)? If we restrict ourselves to spin measurements along z for all three particles, it is evident from (4) that this state is modelled by the Borromean rings with a probability of 1/3 and by the three-Hopf rings with a probability of 2/3. For the state (4) (and the spin measurements considered) it turns out to be impossible to predict the topology of the entanglement with absolute certainty, as we could for the GHZ state (1).

In addition to the configurations shown in Figures 1 and 2, there is one other simple configuration[6] of three interlinked rings. It is the one shown in Figure 3 where the two outer rings are both linked to the central ring but not to each other. Can one exhibit a three-particle state and a set of spin measurements that are modelled by Figure 3? Indeed, one can; consider the state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\beta_1\rangle|\beta_{2x}\rangle|\beta_{3x}\rangle + |\alpha_1\rangle|\alpha_{2x}\rangle|\alpha_{3x}\rangle) \tag{5}$$

$$= \frac{|\beta_2\rangle}{\sqrt{2}}\left(\frac{|\beta_1\rangle|\beta_{3x}\rangle + |\alpha_1\rangle|\alpha_{3x}\rangle}{\sqrt{2}}\right) + \frac{|\alpha_2\rangle}{\sqrt{2}}\left(\frac{|\beta_1\rangle|\beta_{3x}\rangle - |\alpha_1\rangle|\alpha_{3x}\rangle}{\sqrt{2}}\right) \tag{6}$$

$$= \frac{|\beta_3\rangle}{\sqrt{2}}\left(\frac{|\beta_1\rangle|\beta_{2x}\rangle + |\alpha_1\rangle|\alpha_{2x}\rangle}{\sqrt{2}}\right) + \frac{|\alpha_3\rangle}{\sqrt{2}}\left(\frac{|\beta_1\rangle|\beta_{2x}\rangle - |\alpha_1\rangle|\alpha_{2x}\rangle}{\sqrt{2}}\right) \tag{7}$$

$$= \frac{|\beta_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_{2x}\rangle|\beta_{3x}\rangle + |\alpha_{2x}\rangle|\alpha_{3x}\rangle}{\sqrt{2}}\right) + \frac{|\alpha_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_{2x}\rangle|\beta_{3x}\rangle - |\alpha_{2x}\rangle|\alpha_{3x}\rangle}{\sqrt{2}}\right) \tag{8}$$

where $|\beta_{ix}\rangle = (|\beta_i\rangle + |\alpha_i\rangle)/\sqrt{2}$, $|\alpha_{ix}\rangle = (|\beta_i\rangle - |\alpha_i\rangle)/\sqrt{2}$ and where particle 1 is identified with the central ring in Figure 3 and particles 2 and 3 with the two outer rings. For the measurements $O_1 = |\beta_1\rangle\langle\beta_1| - |\alpha_1\rangle\langle\alpha_1|$, $O_2 = |\beta_2\rangle\langle\beta_2| - |\alpha_2\rangle\langle\alpha_2|$

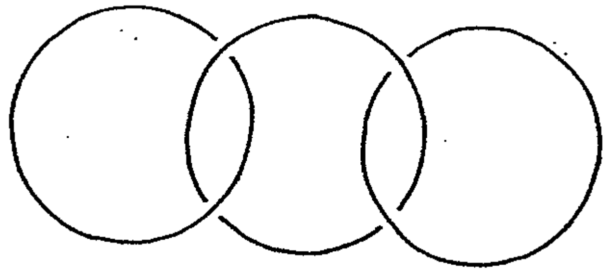

Figure 3. Yet another configuration of three rings. If the central ring is cut, the other two get unlinked; however, if either edge ring is cut, the other two remain linked.

and $O_3 = |\beta_3\rangle\langle\beta_3| - |\alpha_3\rangle\langle\alpha_3|$ on the three particles the above state is modelled by Figure 3 because, as (5), (6) and (7) show, a measurement on particle 1 disentangles the other two particles, whereas a measurement on either particle 2 or 3 leaves the other two particles (maximally) entangled. However, this state exhibits several other topologies as well. Under the measurements $O_1 = |\beta_1\rangle\langle\beta_1| - |\alpha_1\rangle\langle\alpha_1|$, $O_2 = |\beta_{2x}\rangle\langle\beta_{2x}| - |\alpha_{2x}\rangle\langle\alpha_{2x}|$ and $O_3 = |\beta_{3x}\rangle\langle\beta_{3x}| - |\alpha_{3x}\rangle\langle\alpha_{3x}|$, it behaves like the Borromean rings (see Equation (5)), whereas under the measurements $O_1 = |\beta_{1x}\rangle\langle\beta_{1x}| - |\alpha_{1x}\rangle\langle\alpha_{1x}|$, $O_2 = |\beta_2\rangle\langle\beta_2| - |\alpha_2\alpha_2|$ and $O_3 = |\beta_3\rangle\langle\beta_3| - |\alpha_3\rangle\langle\alpha_3|$, it behaves like the three-Hopf rings (see Equations (6), (7) and (8)). More generally, one can ask what happens if one measures the spin of each particle along an arbitrary direction. Then it turns out to be impossible to associate a definite knot configuration with this state; moreover, we cannot generally model it by several alternative knot configurations with a definite probability assigned to each (as we could for the state (4)).

As a last example, consider the N-particle GHZ state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\beta_1\rangle\,|\beta_2\rangle\ldots|\beta_N\rangle - |\alpha_1\rangle\,|\alpha_2\rangle\ldots|\alpha_N\rangle) \tag{9}$$

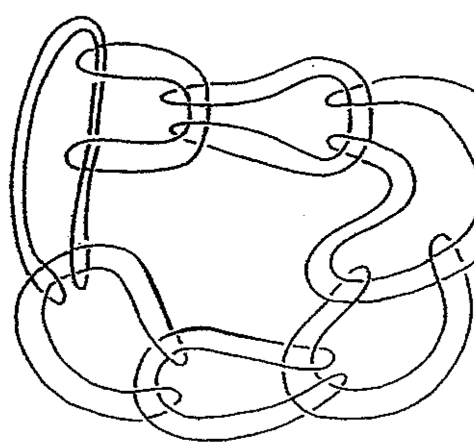

Figure 4. The generalized Borromean rings. These are a set of N interlinked rings that cannot be pulled apart; however, if any one ring is cut, the others can be separated easily.

which is an entangled state of all N particles. However, a measurement of the operator $|\beta_i\rangle\langle\beta_i| - |\alpha_i\rangle\langle\alpha_i|$ on particle i has the effect of disentangling all the remaining particles. Is there a knot configuration that models this situation? Amusingly enough, there is. It is the one shown in Figure 4: one sees that if any of the rings is cut, all the remaining rings are easily separated from each other. The configuration in Figure 4 is termed the generalized Borromean rings in Ref. (7) and it is mentioned as a particular example of a "Brunnian link" in Ref. (8). However, one should add that Figure 4 is not the only topology that one can associate with the state (9). A measurement of the operator $|\beta_i\rangle\langle\alpha_i| + |\alpha_i\rangle\langle\beta_i|$ on particle i throws (9) into a GHZ state of the remaining $N-1$ particles, which is quite different from what we had before. I am not aware of any knot configuration that models this situation and even doubt whether one exists.

What do we learn from the above examples about the connection between entangled quantum states and knot configurations? Do the latter serve in any way to illuminate the former and give us useful ways of thinking about quantum entanglement? In answer to the first question, one must admit that it does not seem possible to develop the analogy between entangled quantum states and knot configurations in any systematic fashion. As noted earlier, a part of the difficulty arises because there is no single quantum process that corresponds to the mathematical act of cutting a knot. A particular quantum state can generally be made to correspond to more than one knot configuration (and sometimes none) by suitably reinterpreting the quantum mechanical process that corresponds to the cutting of a knot. It appears very unlikely that the extensive classification of knot configurations (or links) that has been carried out by mathematicians has any systematic application or utility in the study of entangled quantum states.

Although the analogy between quantum states and knots seems flawed, the attempt to draw parallels between them does suggest some interesting questions. Three of these questions are the following.

(1) Knot or unknot? Given a closed loop of string that is all tangled up, one can ask if it is really knotted or not. If it is unknotted, one can deform it into a circular loop without having to cut and then rejoin any part of it. Analogously, given a multiparticle quantum state, one can ask if it is entangled or not. Even a state that appears horribly entangled may turn out to be a product state in a suitable basis.

(2) Are two knots equivalent? Two knots that appear to be quite different may actually be deformable into one another, in which case they are said to be equivalent. Analogously, two entangled quantum states that appear to be different may simply be the same state written in two alternative bases. How can one tell if this is so?

(3) Which knot is more entangled? Given two knots, one can ask which is more entangled than the other. Analogously, given two multiparticle quantum states, one can ask which has the higher degree of entanglement.

Questions 1 and 2 for knots are difficult topological questions, but they have been answered by mathematicians. The corresponding questions for entangled quantum states are much easier[9] because they involve essentially linear algebra (i.e. basis changes). Question 3 for entangled quantum states was considered recently by Abner Shimony,[10] who showed how to introduce a degree of entanglement for a pair of particles of arbitrary spin. Recently, I[11] have succeeded in generalizing Shimony's measure to an entangled state of arbitrarily many particles, with the entire system being in either a pure or mixed state. My approach involves introducing a hierarchy of entanglement tensors that describe the correlations among any subset of particles in the system. From these tensors one can construct a set of scalars (i.e. quantities invariant under basis changes) that characterize an arbitrary entangled state. These scalars afford one method of tackling questions 1–3 for quantum states posed above. This approach, as well as some of the problems associated with it, will be discussed elsewhere.

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NOTES AND REFERENCES

1 D.M. Greenberger, M.A. Horne and A. Zeilinger, "Going beyond Bell's Theorem", in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, M. Kafatos, ed., Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989, p. 73; D.M. Greenberger, M.A. Horne, A. Shimony and A. Zeilinger, "Bell's Theorem without Inequalities", *Am. J. Phys.* **58**, 1131 (1990); N.D. Mermin, "Quantum Mysteries Revisited", *Am. J. Phys.* **58**, 731 (1990).

2 The z-direction need not be the same for all the particles, but we assume for simplicity that it is.

3 The history of the Borromean rings recounted here has been taken from *Supplement to Not Knot* by D. Epstein and C.Gunn (Jones and Bartlett Publishers, Boston, MA, 1991), p. 7.

4 J.S. Bell, *Physics* (Long Island City, N.Y.) **1**, 195 (1964); J.F. Clauser, M.A. Horne, A.Shimony and R.Holt, *Phys. Rev. Lett.* **23**, 880 (1969).

5 A. Zeilinger, M.A. Horne and D.M. Greenberger, "Higher-order Quantum Entanglement", in *Workshop on Squeezed States and Uncertainty Relations*, D. Han, Y.S. Kim and W.W. Zachary, eds., NASA Conference Publication 3135, 1992.

6 I have in mind only configurations in which each ring is either simply linked or else unlinked with every other ring. By a pair of "simply linked" rings, I mean one for which each ring goes over (and under) the other ring exactly once.

7 L.H. Kauffman, *Knots and Physics* (World Scientific, New Jersey, 1991); Figure 4 of this paper has been taken from p. 38 of this book.

8 C. Livingston, *Knot Theory* (The Mathematical Association of America, Washington D.C., 1993); see exercise 6 on p. 10.

9 In making this remark, I have in mind only finite dimensional quantum systems. For the infinite dimensional case see the essay by Wayne Myrvold in this volume.

10 A. Shimony in *Fundamental Problems in Quantum Theory*, D. Greenberger, ed. (Annals of the New York Academy of Sciences, 1995); A. Shimony, "Measures of Entanglement", presented at the EPR Meeting at the Technion, 20–23 March 1995.

11 P.K. Aravind, unpublished.