Rethinking Quantum States: An Introduction to Q-Bism

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Introduction

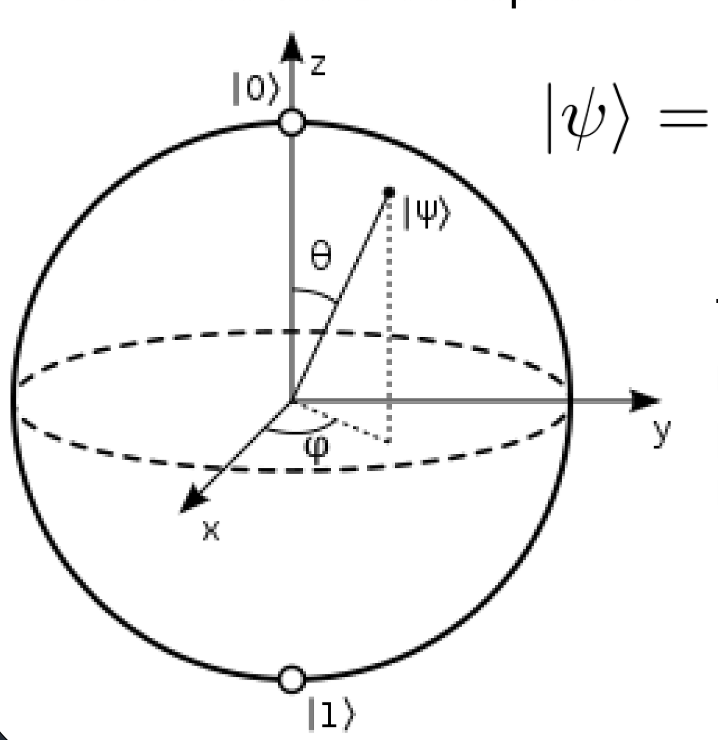
- Many formulations of quantum mechanics that differ from the standard Copenhagen approach have been proposed.
- Quantum Bayesianism (or Q-Bism) is one of them. It originated in the work of Feynman in 1987 and has been further developed by Caves, Fuchs, Schack and others.
- Q-Bism could help resolve some of the interpretational questions about QM that continue to perplex us.

QM: the standard description

- The traditional description of a quantum system is as a state vector that is a linear combination of a number of orthogonal basis vectors :

$$|\psi\rangle = \sum_{n=0}^{d-1} c_n |n\rangle$$

- Any 2-d system (such as a spin-1/2 particle or a photon's polarization) can be represented as a vector on a unit sphere known as the Bloch sphere:



$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi} sin\frac{\theta}{2}|1\rangle$$

The points on the surface represent pure states while the points in the interior represent mixed states

QM: The approach of Q-Bism

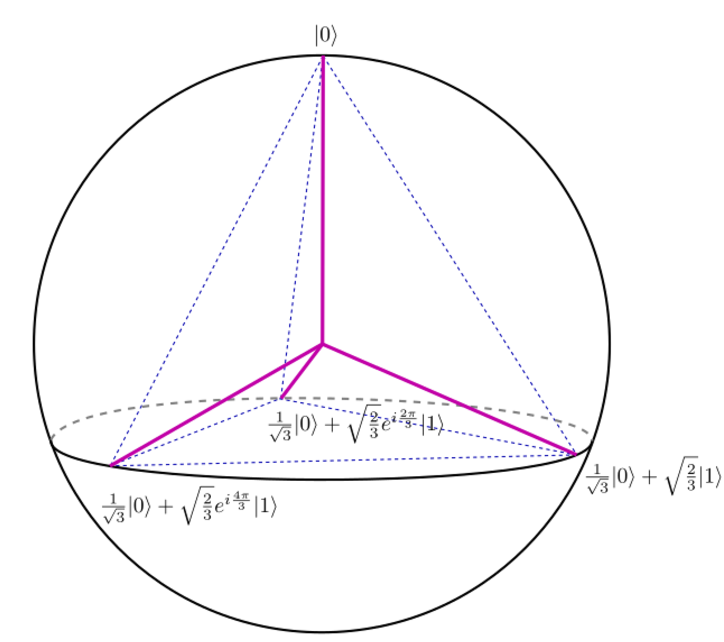
- Q-Bism discards the notion of a wavefunction or a state vector as the starting point for the description of a quantum system.
- Instead, it describes a quantum system as a list of probabilities that describe the outcomes of a particular measurement that can be made on it.
- The list of probabilities, or "probability vector", allows all the properties of the system to be calculated.
- For a d-state quantum system, the probability vector has d^2components that sum to unity.
- **However not every possible probability vector corresponds to a valid state of the system.**

Symmetric Informationally Complete Measurements (SICs)

- SICs are a convenient basis that can be used to generate informationally complete probability vectors.
- In a complex d-dimensional Hilbert space, a set of d^2 unit vectors can be found that satisfy the equation :

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{1}{d+1} \qquad \text{for } j \neq k$$

- For d = 2, the SICs form a regular tetrahedron on the Bloch sphere:



- For d > 2 the SICs form a set of equiangular vectors, but they cannot be easily visualized.

Generating SICs in dimension d

- The so-called Weyl-Heisenberg group can be constructed from two operators X and Z defined via the relations

$$X|j\rangle = |j+1\rangle\,, Z|j\rangle = e^{\frac{2ij\pi}{d}}|j\rangle$$

- By selecting the correct fiducial vector, it is possible to generate a SIC in dimension d by operating on it with all the members of the Weyl-Heisenberg group, given by $X^{\alpha}Z^{\beta}$, where α, β = 0,1,…,d-1.
- Example fiducial vectors in dimension 2 and 3 are:

$$|\psi_0^{(\text{qubit})}\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3+\sqrt{3}} \\ e^{i\pi/4}\sqrt{3-\sqrt{3}}\end{pmatrix} \qquad |\psi_0^{(\text{Hesse})}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ 1 \\ -1\end{pmatrix}$$

- Methods of constructing SICs in many dimensions are known, but a proof of their existence in arbitrary dimensions still eludes us.
- SICs have applications to quantum state tomography and signal analysis.

Constraints on Probability Vectors

- Although the SIC probability vector is a valid representation of a quantum system, not all valid probability vectors represent physically realizable states.
- Beyond the normalization condition, the constraints on the pure-state probability vector are:

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \qquad \sum_{ijk} c_{ijk}\, p(i)p(j)p(k) = \frac{d+7}{(d+1)^3}$$

$$c_{ijk} = \mathrm{Re\,tr}(\Pi_i \Pi_j \Pi_k).$$

where Π_i is the outer product of a SIC state ψ_i with itself.

- This MQP explored the above conditions and their generalization to mixed states in d = 2 and 3.
- In d dimensions a pure state has 2d -2 parameters and a mixed state has $d^2 - 1$. Seeing how these features are accommodated in Q-Bism was another focus of this project.
- Studying SICs and their applications was a third focus.

Feynman's Negative Probabilities

- In Feynman's approach, a spin ½ particle is allowed to have definite spin components along the z- and x-axes simultaneously, which is not allowed in quantum mechanics.
- The most general state of a spin ½ particle is one in which it has probabilities $f_{++}, f_{+-}, f_{-+}, f_{--}$ of being found in the four possible spin states along the z- and x- axes.
- Feynman shows that if these probabilities obey the conditions

$$f_{++} + f_{+-} + f_{-+} + f_{--} = 1, \quad f_{++}^2 + f_{+-}^2 + f_{-+}^2 + f_{--}^2 \leq 1/2,$$

then all calculations done with them according to the usual rules of probability theory lead to the correct results. This is also true for systems of qubits, if each is described in this way. The only catch is that negative probabilities are found at intermediate steps of the calculation, but only for states that can never be observed.
- Q-Bism is a modification of Feynman's approach that avoids negative probabilities and also incorporates ideas from the Bayesian approach to probability.

References

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