Balancing Domain Decomposition Methods for Discontinuous Galerkin Discretization

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Summary. A discontinuous Galerkin (DG) discretization of a Dirichlet problem for second order elliptic equations with discontinuous coefficients in two dimensions is considered. The problem is considered in a polygonal region Ω which is a union of disjoint polygonal substructures Ω_i of size $O(H_i)$. Inside each substructure Ω_i, a triangulation $\mathcal{T}_{h_i}(\Omega_i)$ with a parameter h_i and a conforming finite element method are introduced. To handle nonmatching meshes across $\partial\Omega_i$, a DG method that uses symmetrized interior penalty terms on the boundaries $\partial\Omega_i$ is considered. In this paper we design and analyze Balancing Domain Decomposition (BDD) algorithms for solving the resulting discrete systems. Under certain assumptions on the coefficients and the mesh sizes across $\partial\Omega_i$, a condition number estimate $C(1 + \max_i \log^2 \frac{H_i}{h_i})$ is established with C independent of h_i, H_i and the jumps of the coefficients. The algorithm is well suited for parallel computations and can be straightforwardly extended to three-dimensional problems. Results of numerical tests are included which confirm the theoretical results and the imposed assumption.

1 Introduction

DG methods are becoming more and more popular for approximation of PDEs since they are well suited for dealing with complex geometries, discontinuous coefficients and local or patch refinements; see [2, 4] and the references therein. There are also several papers devoted to algorithms for solving DG discrete problems. In particular in connection with domain decomposition methods, we can mention [9, 10, 1] where overlapping Schwarz methods were proposed and analyzed for DG discretization of elliptic problems with continuous coefficients. In [4] a non optimal multilevel additive Schwarz method is designed and analyzed for the discontinuous coefficient case. In [3] a two-level ASM is proposed and analyzed for DG discretization of fourth order

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problems. In those works, the coarse problems are based on polynomial coarse basis functions on a coarse triangulation. In addition, ideas of iterative substructuring methods and notions of discrete harmonic extensions are not explored, therefore, for the cases where the distribution of the coefficients ρ_i is not quasimonotonic, see [7], these methods when extended straightforwardly to 3-D problems have condition number estimates which might deteriorate as the jumps of the coefficients get more severe. To the best of our knowledge [5] is the only work in the literature that deals with iterative substructuring methods for DG discretizations with discontinuous coefficients, where we have successfully introduced and analyzed BDDC methods with different possible constraints on the edges. A goal of this paper is to design and analyze BDD algorithms, see [11, 8] and also [12], for DG discrete systems with discontinuous coefficients.

The paper is organized as follows. In Section 2, the differential problem and its DG discretization are formulated. In Section 3, the problem is reduced to a Schur complement problem with respect to the unknowns on $\partial\Omega_i$, and discrete harmonic functions defined in a special way are introduced. In Section 4, the BDD algorithm is designed and analyzed. The local problems are defined on $\partial\Omega_i$ and on faces of $\partial\Omega_j$ common to Ω_i, while the coarse space, restriction and prolongation operators are defined via a special partitioning of unity on the $\partial\Omega_i$. Sections 5 and 6 are devoted to numerical experiments and final remarks, respectively.

2 Differential and Discrete Problems

Consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \quad \text{for all } v \in H_0^1(\Omega) \tag{1}$$

where $a(u,v) = \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \nabla v dx$ and $f(v) = \int_\Omega f v dx$.
We assume that $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$ and the substructures Ω_i are disjoint shape regular polygonal subregions of diameter $O(H_i)$ that form a geometrically conforming partition of Ω, i.e., for all $i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty, or a common vertex or face of $\partial\Omega_i$ and $\partial\Omega_j$. We assume $f \in L^2(\Omega)$ and for simplicity of presentation let ρ_i be a positive constant, $i = 1, \dots, N$.

Let us introduce a shape regular triangulation in each Ω_i with triangular elements and the mesh parameter h_i . The resulting triangulation on Ω is in general nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be a finite element (FE) space of piecewise linear continuous functions in Ω_i. Note that we do not assume that the functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Define

$$X_h(\Omega) = X_1(\Omega_1) \times \cdots \times X_N(\Omega_N).$$

The discrete problem obtained by the DG method, see [2, 4], is of the form: Find $u_h^* \in X_h(\Omega)$ such that

$$a_h(u_h^*, v) = f(v) \quad \text{for all } v \in X_h(\Omega) \tag{2}$$

where

$$a_h(u,v) \equiv \sum_{i=1}^{N} b_i(u,v) \quad \text{and} \quad f(v) \equiv \sum_{i=1}^{N} \int_{\Omega_i} f v_i dx, \tag{3}$$

$$b_i(u,v) \equiv a_i(u,v) + s_i(u,v) + p_i(u,v), \tag{4}$$

$$a_i(u,v) \equiv \int_{\Omega_i} \rho_i \nabla u_i \nabla v_i dx, \tag{5}$$

$$s_i(u,v) \equiv \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial u_i}{\partial n}(v_j - v_i) + \frac{\partial v_i}{\partial n}(u_j - u_i) \right) ds, \tag{6}$$

$$p_i(u,v) \equiv \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} (u_j - u_i)(v_j - v_i) ds, \tag{7}$$

$$d_i(u,v) \equiv a_i(u,v) + p_i(u,v), \tag{8}$$

with $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and $v = \{v_i\}_{i=1}^N \in X_h(\Omega)$. We set $l_{ij} = 2$ when $F_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j$ is a common face of $\partial\Omega_i$ and $\partial\Omega_j$, and define $\rho_{ij} = 2\rho_i\rho_j/(\rho_i + \rho_j)$ as the harmonic average of ρ_i and ρ_j, and $h_{ij} = 2h_ih_j/(h_i + h_j)$. In order to simplify the notation we include the index $j = 0$ and set $l_{i0} = 1$ when $F_{i0} \equiv \partial\Omega_i \cap \partial\Omega$ has a positive measure, and set $u_0 = 0$ and $v_0 = 0$, and define $\rho_{i0} = \rho_i$ and $h_{i0} = h_i$. The outward normal derivative on $\partial\Omega_i$ is denoted by $\frac{\partial}{\partial n}$ and δ is the positive penalty parameter.

It is known that there exists a $\delta_0 = O(1) > 0$ such that for $\delta > \delta_0$, we obtain $2|s_i(u,u)| < d_i(u,u)$ and therefore, the problem (2) is elliptic and has a unique solution. An error bound of this method is given in [2] for continuous and in [4, 5] for discontinuous coefficients.

3 Schur Complement Problem

In this section we derive a Schur complement problem for the problem (2).

Define $\overset{o}{X}_i(\Omega_i)$ as the subspace of $X_i(\Omega_i)$ of functions that vanish on $\partial\Omega_i$. Let $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$. For each $i = 1, \ldots, N$, the function $u_i \in X_i(\Omega)$ can be represented as

$$u_i = \hat{\mathcal{P}}_i u + \hat{\mathcal{H}}_i u, \tag{9}$$

where $\hat{\mathcal{P}}_i u$ is the projection of u into $\overset{o}{X}_i(\Omega_i)$ in the sense of $b_i(.,.)$. Note that since $\hat{\mathcal{P}}_i u$ and v_i belong to $\overset{o}{X}_i(\Omega_i)$, we have

$$a_i(\hat{\mathcal{P}}_i u, v_i) = b_i(\hat{\mathcal{P}}_i u, v_i) = a_h(u, v_i). \tag{10}$$

The $\hat{\mathcal{H}}_i u$ is the discrete harmonic part of u in the sense of $b_i(.,.)$, where $\hat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$b_i(\hat{\mathcal{H}}_i u, v_i) = 0 \qquad v_i \in \overset{o}{X}_i(\Omega_i), \tag{11}$$

with boundary data given by

$$u_i \text{ on } \partial\Omega_i \quad \text{and} \quad u_j \text{ on } F_{ji} = \partial\Omega_i \cap \partial\Omega_j. \tag{12}$$

We point out that for $v_i \in \overset{o}{X}_i(\Omega_i)$ we have

$$b_i(\hat{\mathcal{H}}_i u, v_i) = (\rho_i \nabla \hat{\mathcal{H}}_i u, \nabla v_i)_{L^2(\Omega_i)} + \sum_{F_{ij}\subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}. \qquad (13)$$

Note that $\hat{\mathcal{H}}_i u$ is the classical discrete harmonic except at nodal points close to $\partial\Omega_i$. We will sometimes call $\hat{\mathcal{H}}_i u$ by discrete harmonic in a special sense, i.e., in the sense of $b_i(.,.)$ or $\hat{\mathcal{H}}_i$. Hence, $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ and $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u\}_{i=1}^N$ are orthogonal in the sense of $a_h(.,.)$. The discrete solution of (2) can be decomposed as $u_h^* = \hat{\mathcal{P}}u_h^* + \hat{\mathcal{H}}u_h^*$ where for all $v \in X_h(\Omega)$, $a_h(\hat{\mathcal{P}}u_h^*, \hat{\mathcal{P}}v) = f(\hat{\mathcal{P}}v)$ and

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v) = f(\hat{\mathcal{H}}v). \qquad (14)$$

Define $\Gamma \equiv (\cup_i \partial\Omega_{ih_i})$ where $\partial\Omega_{ih_i}$ is the set of nodal points of $\partial\Omega_i$. We note that the nodes on both side of $\cup_i\partial\Omega_i$ belong to Γ. We denote the space $V = V_h(\Gamma)$ as the set of all functions v_h in $X_h(\Omega)$ such that $\hat{\mathcal{P}}v_h = 0$, i.e., the space of discrete harmonic functions in the sense of $\hat{\mathcal{H}}_i$. The equation (14) is the Schur complement problem associated to (2).

4 Balancing Domain Decomposition

We design and analyze a BDD method [11, 12] for solving (14) and use the general framework of balancing domain decomposition methods; see [12]. For $i = 1, \ldots, N$, let V_i be auxiliary spaces and I_i prolongation operators from V_i to V, and define the operators $\tilde{T}_i : V \to V_i$ as

$$b_i(\tilde{T}_i u, v) = a_h(u, I_i v) \quad \text{for all } v \in V_i.$$

and set $T_i = I_i\tilde{T}_i$. The coarse problem is defined as

$$a_h(P_0 u, v) = a_h(u, v) \quad \text{for all } v \in V_0.$$

Then the BDD method is defined as

$$T = P_0 + (I - P_0)\left(\sum_{i=1}^N T_i\right)(I - P_0). \qquad (15)$$

We next define the prolongation operators I_i and the local spaces V_i for $i = 1, ..., N$, and the coarse space V_0. The bilinear forms b_i and a_h are given by (4) and (3), respectively.

4.1 Local Problems

Let us denote by Γ_i the set of all nodes on $\partial\Omega_i$ and on neighboring faces $\bar{F}_{ji} \subset \partial\Omega_j$. We note that the nodes of ∂F_{ji} (which are vertices of Ω_j) are included in Γ_i. Define V_i as the vector space associated to the nodal values on Γ_i and extended via $\hat{\mathcal{H}}_i$ inside Ω_i. We say that $u \in V_i$ if it can be represented as $u := \{u_l^{(i)}\}_{l\in\#(i)}$, where $\#(i) = \{i \text{ and } \cup j : F_{ij} \subset \partial\Omega_i\}$. Here $u_i^{(i)}$ and $u_j^{(i)}$ stand for the nodal value of u on $\partial\Omega_i$ and $\bar{F}_{ji}$. We write $u = \{u_l^{(i)}\} \in V_i$ to refer to a function defined on Γ_i, and $u = \{u_i\} \in V$ to refer to a function defined on all Γ. Let us define the regular zero extension operator $\tilde{I}_i : V_i \to V$ as follows: Given $u \in V_i$, let $\tilde{I}_i u$ be equal to u on the nodes of Γ_i and zero on the nodes of $\Gamma \backslash \Gamma_i$. Then we associate with each $\Omega_k,\ k = 1, \cdots, N$, the discrete harmonic function u_k inside each Ω_k in the sense of $\hat{\mathcal{H}}_k$.

A face across Ω_i and Ω_j has two sides, the side inside $\bar{\Omega}_i$, denoted by F_{ij}, and the side inside $\bar{\Omega}_j$, denoted by F_{ji}. In addition, we assign to each face one master side $m(i,j) \in \{i,j\}$ and one slave side $s(i,j) \in \{i,j\}$. Then, using the *interface condition*, see below, we show that Theorem 1 holds, see below, with a constant C independent of the ρ_i, h_i and H_i.

The Interface Condition. We say that the coefficients $\{\rho_i\}$ and the local mesh sizes $\{h_i\}$ satisfy the *interface condition* if there exist constants C_0 and C_1, of order $O(1)$, such that for any face $F_{ij} = F_{ji}$ the following condition holds

$$h_{s(i,j)} \le C_0 h_{m(i,j)} \quad \text{and} \quad \rho_{s(i,j)} \le C_1 \rho_{m(i,j)}. \tag{16}$$

We associate with each $\Omega_i,\ i = 1, \cdots, N$, the weighting diagonal matrices $D^{(i)} = \{D_l^{(i)}\}_{l \in \#(i)}$ on Γ_i defined as follows:

- On $\partial\Omega_i \ \ (l = i)$

$$D_i^{(i)}(x) = \begin{cases} 1 \text{ if } & x \text{ is a vertex of } \partial\Omega_i, \\ 1 \text{ if } & x \text{ is an interior node of a master face } F_{ij} \\ 0 \text{ if } & x \text{ is an interior node of a slave face } F_{ij} \end{cases} \tag{17}$$

- On $\partial\Omega_j \ \ (l = j)$

$$D_j^{(i)}(x) = \begin{cases} 0 \text{ if } & x \text{ is an end point of } F_{ji}, \\ 1 \text{ if } & x \text{ is an interior node of a slave face } F_{ji} \\ 0 \text{ if } & x \text{ is an interior node of a master face } F_{ji} \end{cases} \tag{18}$$

- For $x \in F_{i0}$ we set $D_i^{(i)}(x) = 1$.

The prolongation operators $I_i : V_i \to V$, $i = 1, \ldots, N$, are defined as $I_i = \tilde{I}_i D^{(i)}$ and they form a partition of unity on Γ described as

$$\sum_{i=1}^{N} I_i \tilde{I}_i^T = I_\Gamma. \tag{19}$$

4.2 Coarse Problem

We define the coarse space $V_0 \subset V$ as

$$V_0 \equiv \mathrm{Span}\{I_i \Phi^{(i)}, i = 1, ..., N\} \tag{20}$$

where $\Phi^{(i)} \in V_i$ denotes the function equal to one at every node of Γ_i.

Theorem 1. *If the interface condition (16) holds then there exists a positive constant C independent of h_i, H_i and the jumps of ρ_i such that*

$$a_h(u,u) \le a_h(Tu,u) \le C(1 + \log^2 \frac{H}{h}) a_h(u,u) \qquad \forall u \in V, \tag{21}$$

where T is defined in (15). Here $\log \frac{H}{h} = \max_i \ \log \frac{H_i}{h_i}$. *(See [6].)*

5 Numerical Experiments

In this section, we present numerical results for the preconditioner introduced in (15) and show that the bounds of Theorem 1 are reflected in the numerical tests. In particular we show that the interface condition (16) is necessary and sufficient.

We consider the domain $\Omega = (0,1)^2$ divided into $N = M \times M$ squares subdomains Ω_i and let $H = 1/M$. Inside each subdomain Ω_i we generate a structured triangulation with n_i subintervals in each coordinate direction and apply the discretization presented in Section 2 with $\delta = 4$. In the numerical experiments we use a red and black checkerboard type of subdomain partition. On the black subdomains we let $n_i = 2 * 2^{L_b}$ and on the red subdomains $n_i = 3 * 2^{L_r}$, where L_b and L_r are integers denoting the number of refinements inside each subdomain Ω_i. Hence, the mesh sizes are $h_b = \frac{2^{-L_b}}{2N}$ and $h_r = \frac{2^{-L_r}}{3N}$, respectively. We consider $-\text{div}(\rho(x)\nabla u^*(x)) = 1$ in Ω with homogeneous Dirichlet boundary conditions. In the numerical experiments we run PCG until the l_2 initial residual is reduced by a factor of 10^6.

In the first test we consider the constant coefficient case $\rho = 1$. We consider different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$, therefore, keeping constant the mesh ratio $h_b/h_r = 3/2$. We place the master on the black subdomains. Table 1 lists the number of PCG iterations and in parenthesis the condition number estimate of the preconditioned system. We note that the interface condition (16) is satisfied. As expected from Theorem 1, the condition numbers appear to be independent of the number of subdomains and grow by a logarithmical factor when the size of the local problems increases. Note that in the case of continuous coefficients the Theorem 1 is valid without any assumption on h_b and h_r if the master sides are chosen on the larger meshes.

Table 1. PCG/BDD iterations count and condition numbers for different sizes of coarse and local problems and constant coefficients ρ_i.

M↓ L_r →	0	1	2	3	4	5
2	13 (6.86)	17 (8.97)	18 (12.12)	19 (16.82)	21 (22.23)	22 (28.25)
4	18 (8.39)	22 (11.30)	26 (14.74)	30 (19.98)	33 (26.64)	36 (34.19)
8	20 (8.89)	24 (11.57)	28 (14.82)	32 (20.03)	37 (26.64)	42 (34.04)
16	19 (9.02)	24 (11.63)	27 (14.83)	32 (20.05)	37 (26.67)	42 (34.06)

We now consider the discontinuous coefficient case where we set $\rho_i = 1$ on the black subdomains and $\rho_i = \mu$ on the red subdomains. The subdomains are kept fixed to 4×4. Table 2 lists the results on runs for different values of μ and for different levels of refinements on the red subdomains. On the black subdomains $n_i = 2$ is kept fixed. The masters are placed on the black subdomains. It is easy to see that the interface condition (16) holds if and only if μ is not large, which it is in agreement with the results in Table 2.

Table 2. PCG/BDD iterations count and condition numbers for different values of the coefficients and the local mesh sizes on the red subdomains only. The coefficients and the local mesh sizes on the black subdomains are kept fixed. The subdomains are also kept fixed to 4×4.

$\mu \downarrow L_r \rightarrow$	0	1	2	3	4
1000	90 (2556)	133 (3744)	184 (5362)	237 (7178)	303 (9102)
10	33 (29.16)	40 (42.31)	47 (58.20)	52 (75.55)	57 (94.59)
0.1	17 (8.28)	19 (8.70)	19 (9.21)	19 (9.50)	19 (9.65)
0.001	18 (8.83)	18 (8.95)	18 (9.46)	18 (9.83)	18 (10.08)

6 Final Remarks

We end this paper by mentioning extensions and alternative Neumann-Neumann methods for DG discretizations where the Theorem 1 holds: 1) The BDD algorithms can be straightforwardly extended to three-dimensional problems; 2) Additive Schwarz versions and inexact local Neumann solvers can be considered; see [6]; 3) On faces F_{ij} where h_i and h_j are of the same order, the values of (17) and (18) at interior nodes x of the faces F_{ij} and F_{ji} can be replaced by $\frac{\sqrt{\rho_i}}{\sqrt{\rho_i}+\sqrt{\rho_j}}$. 4) Similarly, on faces F_{ij} where ρ_i and ρ_j are of the same order, we can replace (17) and (18) at interior nodes x of the faces F_{ij} and F_{ji} by $\frac{h_i}{h_i+h_j}$. Finally, we remark the conditioning of the preconditioned systems deteriorates as we increase the penalty parameter δ to large values.

References

[1] P. F. Antonietti and B. Ayuso. Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: non-overlapping case. Technical Report 20-VP, IMATI-CNR, June 2005. To appear in M2AN.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and D. Martin. Unified analysis of discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[3] S. Brenner and K. Wang. Two-level additive Schwarz preconditioners for C^0 interior penalty methods. *Numer. Math.*, 102(2):231–255, 2005.

[4] M. Dryja. On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients. *Comput. Methods Appl. Math.*, 3(1):76–85, 2003.

[5] M. Dryja, J. Galvis, and M. Sarkis. BDDC methods for discontinuous Galerkin discretization of elliptic problems. *J. Complexity*, 2007. To appear.

[6] M. Dryja, J. Galvis, and M. Sarkis. Neumann-Neumann methods for discontinuous Galerkin discretization of elliptic problems, 2007. In preparation.

[7] M. Dryja, M. Sarkis, and O. B. Widlund. Multilevel Schwarz methods for elliptic problems with discontinuous coefficients in three dimensions. *Numer. Math.*, 72(3):313–348, 1996.

[8] M. Dryja and O. B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite elements problems. *Comm. Pure Appl. Math.*, 48(2):121–155, 1995.

[9] X. Feng and O. A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365, 2001.
[10] C. Lasser and A. Toselli. An overlapping domain decomposition preconditioners for a class of discontinuous Galerkin approximations of advection-diffusion problems. *Math. Comp.*, 72(243):1215–1238, 2003.
[11] J. Mandel. Balancing domain decomposition. *Comm. Numer. Methods Engrg.*, 9(3):233–241, 1993.
[12] A. Toselli and O.B. Widlund. *Domain Decomposition Methods—Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, 2005.