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Numerical boundary corrector for elliptic equations with rapidly oscillating periodic coefficients

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SUMMARY

We develop a numerical method to solve

$$L_\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i} a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} u_\varepsilon = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega$$

where the matrix $a(y) = (a_{ij}(y))$ is symmetric positive definite, whose entries are periodic functions of y with periodic cell Y. More specifically we assume $a_{ij} \in C^{1,\beta}(\Re^2)$, $\beta > 0$. It is also assumed that there exists positive constants γ_a and β_a such that $\gamma_a \|\xi\|^2 \leqslant a_{ij}(y)\xi_i\xi_j \leqslant \beta_a \|\xi\|^2$ for all $\xi \in \Re^2$ and $y \in \bar{Y}$. The major goal in this paper is to develop a numerical approximation scheme on a mesh size $h > \varepsilon$ (or $h \gg \varepsilon$) with quasi-optimal approximation on L^2 and broken H^1 norms. The new method is based on asymptotic analysis and a careful treatment of the boundary corrector term. This kind of equation has applications in areas such as the study of flow through porous media and composite materials.

KEY WORDS: finite elements; homogenization; elliptic equations; multi-scaling

1. INTRODUCTION

On several real world problems the scale ε is so much smaller than Ω that even with very heavy computer efforts it is impossible to take $h < \varepsilon$, h being the scale (mesh-size) of the

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discrete method used to approximate the solution of

$$L_\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i} a_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j} u_\varepsilon = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1}$$

The major goal in this article is to develop a numerical scheme on a mesh size $h > \varepsilon$ (or $h \gg \varepsilon$). We note that when $h > \varepsilon$ standard finite element methods do not result in good numerical approximations; see Reference [1].

One of the first mathematical tools used to attack this problem was the homogenization theory [2, 3]. Based on this theory, we consider a first-order expansion of u_ε plus a boundary corrector term and then we separately approximate each term numerically. The original part of this paper is on the design of numerical boundary correctors. The construction of boundary correctors that are suitable for numerical approximation is a key issue in this work.

Recently, new numerical methods have been proposed for solving this problem, such as, the multi-scale finite element methods [4–6], the residual-free bubble function methods [7–9], and the generalized FEM for homogenization problems [10]. There are also related methods in case the homogenized equation is not known; see References [11, 12]. The method proposed here, as opposed to the methods [5–9], is strongly based on asymptotic expansions of u_ε.

2. NOTATION

We assume that $\Omega = Y = [0,1] \times [0,1]$, and introduce the following notation:

$$\Gamma_e = \{x_1 = 1, x_2 \in [0,1]\}, \quad \Gamma_w = \{x_1 = 0, x_2 \in [0,1]\}$$

$$\Gamma_n = \{x_2 = 1, x_1 \in [0,1]\}, \quad \Gamma_s = \{x_2 = 0, x_1 \in [0,1]\}$$

where Γ_k, $k \in \{e, w, n, s\}$ denotes a generic side of $\partial\Omega$.

Let $D \subset \Re^2$ be an open set. We use the standard notation $\|\cdot\|_{s,D}$, $\|\cdot\|_{s,p,D}$ for $H^s(D)$ and $W^s_p(D)$ norms, and $|\cdot|_{s,D}$, $|\cdot|_{s,p,D}$ their semi-norms. We also define broken norms by

$$\|v\|_{s,h,D} = \sqrt{\sum_{K_j \in \mathscr{T}_h(D)} \|v\|^2_{H^s(K_j)}}$$

where $\mathscr{T}_h(D) = K_1, K_2, \ldots, K_m$ is a regular partition of D with size h. Throughout this paper, when we do not make reference to the domain D it is assumed that $D = \Omega$ or Y. It continually uses the Einstein summation convention, i.e. repeated indices indicate summation. In what follows c denotes a generic constant independent of ε, h, and functions being evaluated.

3. THEORETICAL APPROXIMATION

3.1. The asymptotic expansion

The solution u_ε can be approximated by an asymptotic expansion. This approximation can be found using Equation (1) and the ansatz

$$u_\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots$$

where the functions $u_j(x,y)$ are Y periodic in y. These terms are defined below; for more details see References [3, 13, 14].

Let χ^j be the Y periodic solution with zero average on Y of

$$\nabla_y \cdot a(y)\nabla_y \chi^j = \nabla_y \cdot a(y)\nabla_y y_j = \frac{\partial}{\partial y_i} a_{ij}(y) \tag{2}$$

We have that $\chi^j \in C^{2,\beta}(\Re^2)$ when $a_{ij} \in C^{1,\beta}(\Re^2)$; see Theorem 12.1 from Reference [15]. Define the matrix

$$A_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\,\frac{\partial}{\partial y_l}(y_i - \chi^i)\,\frac{\partial}{\partial y_m}(y_j - \chi^j)\,\mathrm{d}y \tag{3}$$

It is easy to see that the matrix A is symmetric positive definite. Define $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ as the solution of

$$-\nabla . A\nabla u_0 = f \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial\Omega \tag{4}$$

and let

$$u_1\left(x, \frac{x}{\varepsilon}\right) = -\chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$$

Note that $u_0 + \varepsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$. In order to correct this, the boundary corrector term $\theta_\varepsilon \in H^1(\Omega)$ is introduced as the solution of

$$-\nabla \cdot a(x/\varepsilon)\nabla\theta_\varepsilon = 0 \quad \text{in } \Omega, \quad \theta_\varepsilon = -u_1\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega \tag{5}$$

Therefore we obtain $u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon \in H_0^1(\Omega)$ and it can be shown [13] that the following estimates hold:

$$\|u_\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon)\|_0 \leqslant c\varepsilon^2\|u_0\|_3$$

and

$$\|u_\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon)\|_1 \leqslant c\varepsilon\|u_0\|_2$$

3.2. Boundary corrector approximation

Note that the coefficients $a_{ij}(x/\varepsilon)$ and the boundary values $-u_1(x, x/\varepsilon)$ of Equation (5) are highly oscillatory, hence it is not a trivial problem to obtain a good discretization for θ_ε. We propose an analytical approximation for θ_ε, denoted by ϕ_ε that satisfies the oscillating boundary condition and is more suitable for numerical approximation.

Note that $u_0 = 0$ along $\partial\Omega$ implies $\nabla u_\varepsilon|_{\partial\Omega} = \eta\partial_\eta u_0$, where η denotes the unity outward normal vector on $\partial\Omega$ and $\partial_\eta u_0$ denotes the unity outward derivative of u_0 (see Remark 3.1). We then decompose $\theta_\varepsilon = \tilde{\theta}_\varepsilon + \bar{\theta}_\varepsilon$ where

$$-\nabla \cdot a(x/\varepsilon)\nabla\tilde{\theta}_\varepsilon = 0 \quad \text{in } \Omega, \quad \tilde{\theta}_\varepsilon = -u_1 - \chi^*\partial_\eta u_0 = \left(\chi^j\left(\frac{x}{\varepsilon}\right)\eta_j - \chi^*\right)\partial_\eta u_0 \quad \text{on } \partial\Omega \tag{6}$$

and

$$-\nabla \cdot a(x/\varepsilon)\nabla\bar{\theta}_\varepsilon = 0 \quad \text{in } \Omega, \quad \bar{\theta}_\varepsilon = \chi^*\partial_\eta u_0 \quad \text{on } \partial\Omega \tag{7}$$

where $\chi^*|_{\Gamma_k} = \chi_k^*$ are properly chosen constants. In Remark 3.1 we show that $\partial_\eta u_0|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$, hence $\chi^* \partial_\eta u_0 \in H^{1/2}(\partial\Omega)$, and therefore problems (6) and (7) are well posed. The approximation ϕ_ε for θ_ε is defined later as $\tilde{\phi}_\varepsilon + \bar{\phi}_\varepsilon$, where $\tilde{\phi}_\varepsilon \approx \tilde{\theta}_\varepsilon$ and $\bar{\phi}_\varepsilon \approx \bar{\theta}_\varepsilon$.

Next we define constants χ_k^* for which the approximation $\tilde{\phi}_\varepsilon$ decays exponentially to zero away from the boundary and is suitable for numerical approximation. Also $\tilde{\phi}_\varepsilon$ satisfies the correct Dirichlet condition $-u_1(x, x/\varepsilon) - \chi^* \partial_\eta u_0$ on $\partial\Omega$.

3.2.1. Calculating the constants χ_k^*. Associated to each side of Ω define the functions v_k, $k \in \{e, w, n, s\}$, as

1. Let $G_e = \{(-\infty, 0] \times [0, 1]\}$ and v_e the solution of

$$
\begin{aligned}
-\nabla_y \cdot a(y_1, y_2) \nabla_y v_e = 0 \quad & \text{in } G_e \\
v_e(0, y_2) = \chi^1(1, y_2) \quad & \text{for } 0 < y_2 < 1 \\
v_e(y_1, 0) = v_e(y_1, 1) \quad & \text{for } -\infty < y_1 < 0
\end{aligned}
$$

   and

$$
\frac{\partial v_e}{\partial y_i} \exp(-\gamma y_1) \in L^2(G_e), \quad i = 1, 2
$$

2. Let $G_w = \{[0, \infty) \times [0, 1]\}$ and v_w the solution of

$$
\begin{aligned}
-\nabla_y \cdot a(y_1, y_2) \nabla_y v_w = 0 \quad & \text{in } G_w \\
v_w(0, y_2) = -\chi^1(1, y_2) \quad & \text{for } 0 < y_2 < 1 \\
v_w(y_1, 0) = v_w(y_1, 1) \quad & \text{for } 0 < y_1 < \infty
\end{aligned}
$$

   and

$$
\frac{\partial v_w}{\partial y_i} \exp(\gamma y_1) \in L^2(G_w), \qquad i = 1, 2
$$

3. Let $G_n = \{[0, 1] \times (-\infty, 0]\}$ and v_n the solution of

$$
\begin{aligned}
-\nabla_y \cdot a(y_1, y_2) \nabla_y v_n = 0 \quad & \text{in } G_n \\
v_n(y_1, 0) = \chi^2(y_1, 1) \quad & \text{for } 0 < y_1 < 1 \\
v_n(0, y_2) = v_n(1, y_2) \quad & \text{for } -\infty < y_2 < 0
\end{aligned}
$$

   and

$$
\frac{\partial v_n}{\partial y_i} \exp(-\gamma y_2) \in L^2(G_n), \quad i = 1, 2
$$

4. Let $G_s = \{[0,1] \times [0,\infty)\}$ and v_s the solution of

$$
\begin{aligned}
-\nabla_y \cdot a(y_1, y_2)\nabla_y v_s &= 0 \quad \text{in } G_s \\
v_s(y_1, 0) &= -\chi^2(y_1, 0) \quad \text{for } 0 < y_1 < 1 \\
v_s(0, y_2) &= v_s(1, y_2) \quad \text{for } 0 < y_2 < \infty
\end{aligned}
$$

and

$$\frac{\partial v_s}{\partial y_i} \exp(\gamma y_2) \in L^2(G_s), \quad i = 1, 2$$

From Reference [13, Section 6] there exists a unique solution for each of the above equations. Let

$$
\begin{aligned}
\chi_k^* = {}& \frac{1}{(A\eta^k, \eta^k)} \int_{\Gamma_k} \left[\chi^l a_{ij} \left(\delta_{jm} - \frac{\partial \chi^m}{\partial y_j} \right) \eta_i^k \eta_m^k \eta_l^k \right] \bigg|_{\Gamma_k} \mathrm{d}s \\
& + \int_{G_k} (a(y_1, y_2)\nabla_y v_k \cdot \nabla_y v_k)\, \mathrm{d}y
\end{aligned}
$$

where η^k denotes the unity outward normal at Γ_k and η_i^k its ith component. It can be shown [13] that $v_e - \chi_e^*$ decays exponentially to zero for $y_1 \to -\infty$, i.e.

$$(v_e - \chi_e^*) \exp(-\gamma y_1) \in L^2(G_e)$$

Similar results also hold when $k \in \{w, n, s\}$.

3.2.2. Approximating $\tilde{\theta}_\varepsilon$. We note by Remark 3.1 that $(u_1(x, x/\varepsilon) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$. Thus we can split $\tilde{\theta}_\varepsilon = \sum_{k \in \{e,w,n,s\}} \tilde{\theta}_\varepsilon^k$, where

$$L_\varepsilon \tilde{\theta}_\varepsilon^k = 0 \quad \text{in } \Omega, \quad \tilde{\theta}_\varepsilon^k = \begin{cases} -u_1\left(x, \dfrac{x}{\varepsilon}\right) - \chi^* \partial_\eta u_0 & \text{on } \Gamma_k \\ 0 & \text{on } \partial\Omega \setminus \Gamma_k \end{cases}$$

We approximate $\tilde{\theta}_\varepsilon^k$ by $\tilde{\phi}_\varepsilon^k$, given by

$$
\begin{aligned}
\tilde{\phi}_\varepsilon^e(x_1, x_2) &= \varphi_e(x_1) \left(v_e\left(\frac{x_1 - 1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) - \chi_e^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2) \\
\tilde{\phi}_\varepsilon^w(x_1, x_2) &= -\varphi_w(x_1) \left(v_w\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) - \chi_w^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2) \\
\tilde{\phi}_\varepsilon^n(x_1, x_2) &= \varphi_n(x_2) \left(v_n\left(\frac{x_1}{\varepsilon}, \frac{x_2 - 1}{\varepsilon} \right) - \chi_n^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2) \\
\tilde{\phi}_\varepsilon^s(x_1, x_2) &= -\varphi_s(x_2) \left(v_s\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) - \chi_s^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2)
\end{aligned}
\tag{8}
$$

where φ_k are non-negative smooth functions satisfying

$$\varphi_e(s) = \varphi_n(s) = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{if } s = 0, \end{cases} \qquad \varphi_w(s) = \varphi_s(s) = \begin{cases} 0 & \text{if } s = 1 \\ 1 & \text{if } s = 0 \end{cases}$$

Hence

$$\tilde{\phi}_\varepsilon = \sum_{k \in \{e,w,n,s\}} \tilde{\phi}_\varepsilon^k$$

approximate $\tilde{\theta}_\varepsilon$, and $\tilde{\phi}_\varepsilon = \tilde{\theta}_\varepsilon$ on the boundary of Ω.

3.2.3. Approximating $\bar{\theta}_\varepsilon$. The boundary condition imposed on Equation (7) does not depend on ε. An effective approximation for $\bar{\theta}_\varepsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the solution of

$$-\nabla \cdot A \nabla \bar{\phi} = 0 \quad \text{in } \Omega, \quad \bar{\phi} = \chi^* \partial_\eta u_0 \quad \text{on } \partial\Omega$$

We define our theoretical approximation for u_ε as $u_0 + \varepsilon u_1 + \varepsilon \phi_\varepsilon$, where

$$\phi_\varepsilon = \tilde{\phi}_\varepsilon + \bar{\phi}$$

Note that $\phi_\varepsilon|_{\partial\Omega} = \theta_\varepsilon|_{\partial\Omega}$, therefore $u_0 + \varepsilon u_1 + \varepsilon \phi_\varepsilon = 0$ on $\partial\Omega$. In Reference [16] we prove the following error bounds.

Theorem 3.1
Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in H^2(\Omega)$. Then there exists a constant c, such that

$$\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon \phi_\varepsilon\|_1 \leqslant c\varepsilon \|u_0\|_2$$

Theorem 3.2
Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in H^3(\Omega)$. Then there exists a constant c, such that

$$\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon \phi_\varepsilon\|_0 \leqslant c\varepsilon^{3/2} \|u_0\|_3$$

Remark 3.1
In the case $\Omega = [0,1] \times [0,1]$ we have

$$\partial_\eta u_0 = \begin{cases} \dfrac{\partial u_0}{\partial x_1} & \text{on } \Gamma_e \\ -\dfrac{\partial u_0}{\partial x_1} & \text{on } \Gamma_w \\ \dfrac{\partial u_0}{\partial x_2} & \text{on } \Gamma_n \\ -\dfrac{\partial u_0}{\partial x_2} & \text{on } \Gamma_s \end{cases}$$

Since u_0 satisfies zero Dirichlet boundary condition on $\partial\Omega$ and $u_0 \in H^2(\Omega)$, we have $\partial u_0/\partial x_1|_{\Gamma_n \cup \Gamma_s} = 0$ and $\partial u_0/\partial x_2|_{\Gamma_e \cup \Gamma_w} = 0$. Therefore,

$$\partial_\eta u_0 = \left(\varphi_e \frac{\partial u_0}{\partial x_1} - \varphi_w \frac{\partial u_0}{\partial x_1} + \varphi_n \frac{\partial u_0}{\partial x_2} - \varphi_s \frac{\partial u_0}{\partial x_2} \right)\Bigg|_{\partial\Omega}$$

where each term on the right-hand side satisfies $\varphi_k(\partial u_0/\partial x_{j_k}) = 0$ on $\partial\Omega \setminus \Gamma_k$. Using that $\varphi_k(\partial u_0/\partial x_{j_k}) \in H^1(\Omega)$ we obtain $\varphi_k(\partial u_0/\partial x_{j_k})|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$ and

$$\begin{aligned} \|\chi^* \partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} &\leqslant \left\| \varphi_e \chi_e^* \frac{\partial u_0}{\partial x_1} - \varphi_w \chi_w^* \frac{\partial u_0}{\partial x_1} + \varphi_n \chi_n^* \frac{\partial u_0}{\partial x_2} - \varphi_s \chi_s^* \frac{\partial u_0}{\partial x_2} \right\|_1 \\ &\leqslant c(\chi^*)\|u_0\|_2 \end{aligned}$$

Note also that $u_1(x, x/\varepsilon) = -\chi^j(x/\varepsilon)(\partial u_0/\partial x_j)(x)$. Since $\chi^j \in C^{2,\beta}(\Re^2)$ we can use the same argument given in this remark to show that $u_1|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$.

4. FINITE ELEMENT APPROXIMATION

We now give the algorithm to obtain the numerical approximation for u_ε:

Step 1: Solve the cell problem (2) with a second-order accurate conforming finite element in a partition $\mathscr{T}_{\hat{h}}(Y)$. Call these solutions $\chi^j_{\hat{h}}$.

Step 2: Obtain $A^{\hat{h}}$ by

$$A^{\hat{h}}_{ij} = \frac{1}{|Y|} \int_Y a_{lm}(y) \frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}}) \frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})\,\mathrm{d}y$$

Step 3: Define $V^h(\Omega) = \{v \in C^0(\Omega);\ v|_K \in \mathscr{Q}_1(K),\ K \in \mathscr{T}_h(\Omega),\ K \text{ rectangular}\}$ and $V^h_0(\Omega) = V^h(\Omega) \cap H^1_0(\Omega)$. Let $u^{h,\hat{h}}_0 \in V^h_0$ satisfying

$$\int_\Omega (A^{\hat{h}} \nabla u^{h,\hat{h}}_0, \nabla v^h)\,\mathrm{d}x = \int_\Omega f v^h\,\mathrm{d}x \quad \forall v^h \in V^h_0$$

The justification for using a rectangular mesh is postponed to Remark 4.1.

Step 4: Define $u^{h,\hat{h}}_1$ as

$$u^{h,\hat{h}}_1(x) = -\chi^j_{\hat{h}}\left(\frac{x}{\varepsilon}\right) \frac{\partial u^{h,\hat{h}}_0}{\partial x_j}(x)$$

Note that this leads to a non-conforming approximation for u_1 in the partition $\mathscr{T}_h(\Omega)$.

Step 5: Let p be a positive integer and $G^p_e = [-p, 0] \times [0, 1]$. Define $\tilde{v}_e \in H^1(G^p_e)$ the solution of

$$\begin{aligned} -\nabla_y \cdot a(y_1, y_2)\nabla_y \tilde{v}_e &= 0 \quad \text{in } G^p_e \\ \tilde{v}_e(0, y_2) &= \chi^1_{\hat{h}}(1, y_2), \quad 0 \leqslant y_2 \leqslant 1 \\ \partial_\eta \tilde{v}_e &= 0 \quad \text{on } \{y \in G^p_e;\ y_1 = -p\} \end{aligned}$$

and

$$\tilde{v}_e(y_1, 0) = \tilde{v}_e(y_1, 1), \quad -p \leqslant y_1 \leqslant 0$$

Let $v_e^{\hat{h},p}$ be a numerical approximation of $\tilde{v}_e$ using a second-order accurate conforming finite element on a mesh $\mathscr{T}_{\hat{h}}(G_e^p)$.

Step 6: Define

$$\chi_e^{*,\hat{h},p} = \frac{1}{A_{11}^{\hat{h}}} \int_0^1 \left(\chi_{\hat{h}}^1(1, y_2) a_{1k}(1, y_2) \left[\delta_{k1} - \frac{\partial \chi_{\hat{h}}^1(1, y_2)}{\partial y_2} \right] \right) \mathrm{d}y_2$$
$$+ \int_{G_e^p} (a(y_1, y_2) \nabla_y v_e^{\hat{h},p} \cdot \nabla_y v_e^{\hat{h},p}) \, \mathrm{d}y$$

The other cases $k \in \{w, n, s\}$ are treated similarly.

Step 7: Let $\bar{\phi}^{h,\hat{h},p}$ be a second-order accurate finite element approximation in a mesh of size h for the following equation:

$$-\nabla A^{\hat{h}} \nabla \psi = 0, \quad \psi = \chi^{*,\hat{h},p} \partial_\eta u_0^{\hat{h},h} \quad \text{on } \partial\Omega \tag{9}$$

Remark 4.1

Since $u_0^{\hat{h},h} \in H_0^1(\Omega)$, the domain Ω is rectangular, and bilinear rectangular elements are considered to obtain $u_0^{\hat{h},h}$, it is easy to see that $\partial_\eta u_0^{\hat{h},h}$ is continuous on $\partial\Omega$ and linear in every edge of $\mathscr{T}_h(\partial\Omega)$. Observe also that the zero Dirichlet boundary condition implies $\partial_\eta u_0^{\hat{h},h} = 0$ at the corners of Ω. Therefore, $\chi^{*,\hat{h},p} \partial_\eta u_0^{\hat{h},h} \in H^{1/2}(\partial\Omega)$ and Equation (9) is well posed. Taking $\bar{\phi}^{h,\hat{h},p} \in V^h$ allows us to use the same stiffness matrix for obtaining $u_0^{\hat{h},h}$ and $\bar{\phi}^{h,\hat{h},p}$.

Step 8: Observe that in Equation (8) the term $v_e((x_1 - 1)/\varepsilon, x_2/\varepsilon)$ appears. Since the approximation $v_e^{\hat{h},p}$ is defined in G_e^p, we can calculate $v_e^{\hat{h},p}((x_1 - 1)/\varepsilon, x_2/\varepsilon)$ only if $x_1 \geqslant 1 - \varepsilon p$. Since the functions $v_k - \chi_k^*$ decays exponentially to zero away from the boundary, it is natural to consider the following approximation:

$$\tilde{\phi}_\varepsilon^{e,h,\hat{h},p}(x_1, x_2) = \begin{cases} \varphi_e(x_1) \left(v_e^{\hat{h},p} \left(\dfrac{x_1 - 1}{\varepsilon}, \dfrac{x_2}{\varepsilon} \right) - \chi_e^{*,\hat{h},p} \right) \dfrac{\partial u_0^{h,\hat{h}}}{\partial x_1} & \text{if } x_1 > 1 - \varepsilon p \\ 0 & \text{if } x_1 \leqslant 1 - \varepsilon p \end{cases} \tag{10}$$

and

$$\tilde{\phi}_\varepsilon^{h,\hat{h},p} = \sum_{k \in \{e,w,n,s\}} \tilde{\phi}_\varepsilon^{k,h,\hat{h},p}$$

Step 9: Approximate θ_ε by $\phi_\varepsilon^{h,\hat{h},p} = \tilde{\phi}_\varepsilon^{h,\hat{h},p} + \bar{\phi}^{h,\hat{h},p}$ and finally construct the numerical approximation for u_ε as

$$u_\varepsilon^{h,\hat{h},p} = u_0^{h,\hat{h}} + \varepsilon u_1^{h,\hat{h}} + \varepsilon \phi_\varepsilon^{h,\hat{h},p}$$

Remark 4.2
Only two stiffness matrices are needed to be formed: one for Steps 3 and 7, and another one for Steps 1 and 5. In Step 5, an iterative method based on vector–matrix multiplication, together with the periodicity of the matrix on Step 1, is explored.

5. ERROR ANALYSIS

When $p \to \infty$ and $\hat{h} \to 0$ we prove in Reference [16] the following estimates.

Theorem 5.1
Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in W^{2,\infty}(\Omega)$. Then there exists a constant c, such that

$$|u_\varepsilon - u_h|_{1,h} \leqslant c(h+\varepsilon)\|u_0\|_{2,\infty}$$

Theorem 5.2
Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$. Then there exists a constant c, such that

$$\|u_\varepsilon - u_h\|_0 \leqslant c(h^2 + \varepsilon^{3/2} + \varepsilon h \ln(h))(|u_0|_{2,\infty} + \|u_0\|_3)$$

6. NUMERICAL EXPERIMENTS

In this section, we present some numerical results for solving our model problem with

$$a(x/\varepsilon) = \left(\frac{2 + P\sin(2\pi x_1/\varepsilon)}{2 + P\cos(2\pi x_2/\varepsilon)} + \frac{2 + \sin(2\pi x_2/\varepsilon)}{2 + P\sin(2\pi x_1/\varepsilon)}\right) I_{2\times 2}$$

$$f(x) = -1 \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \text{ and } P = 1.8$$

We compare the solution obtained by our method with the solution obtained by a second-order accurate finite element method in a fine mesh of size h_f, which we call u_ε^*. Tables I and II provide absolute error estimates for $u_\varepsilon^* - u_\varepsilon^{h,\hat{h},p}$, on the $\|\cdot\|_0$ norm and $|\cdot|_{1,h}$ semi-norm for different values of h and ε. We have used $p = 2$, $\hat{h} = \frac{1}{128}$, $h_f = \frac{1}{2048}$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_e^{\hat{h},p}$.

From Tables I and II, we see that for $\varepsilon \ll h$, we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and semi-norm H^1, respectively. We observe that when we fix h, $\hat{h}$ and p, and decrease ε, the errors almost do not change, hence the dominant error term is $O(h)$. Also looking at the diagonal values in these tables, we see clearly that the numerical error agrees with the theoretical rates from Theorems 5.1 and 5.2.

Table I. $\|\cdot\|_0$ error.

$\varepsilon\downarrow$ $h\to$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{16}$	2.7085e − 04	7.7993e − 05		
$\frac{1}{32}$	2.6300e − 04	6.6246e − 05	1.7773e − 05	
$\frac{1}{64}$	2.5388e − 04	5.9446e − 05	1.4414e − 05	1.2137e − 05

Table II. $|\cdot|_{1,h}$ error.

$\varepsilon\downarrow$ $h\rightarrow$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{16}$	0.0097	0.0066		
$\frac{1}{32}$	0.0089	0.0051	0.0036	
$\frac{1}{64}$	0.0086	0.0045	0.0026	0.0018

Table III. $\varepsilon=\frac{1}{64}$, $h=\frac{1}{32}$, $h_f=\frac{1}{1024}$.

	$\|\cdot\|_0$	$|\cdot|_{1,h}$
$u_\varepsilon^* - u_0^{h,\hat{h}}$	0.0287	0.0215
$u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}}$	0.0213	0.0026
$u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}} - \varepsilon\bar{\phi}^{h,\hat{h},p}$	6.1557e − 05	0.0026
$u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}} - \varepsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}_\varepsilon^{h,\hat{h},p})$	6.1557e − 05	0.0024

Table IV. $p=2$, $\hat{h}=64$.

h	ε	$\|u_\varepsilon^* - u_\varepsilon^{h,\hat{h},p}\|_0$	MsFEM-O L^2 error
$\frac{1}{16}$	$\frac{1}{25}$	6.92e − 05	6,23 e − 05
$\frac{1}{32}$	$\frac{1}{50}$	1.77e − 05	8,43 e − 05
$\frac{1}{64}$	$\frac{1}{100}$	1.24e − 05	9,32 e − 05

Table III shows the improvement obtained in the final approximation by considering the numerical approximation for the boundary corrector. We observe a better improvement on the $\|\cdot\|_0$ norm rather than on the $|\cdot|_{1,h}$ semi-norm. The reason for this is that $\bar{\phi}$ is obtained through the homogenized equation associated with Problem (7), therefore, it is a good approximation for $\bar{\theta}_\varepsilon$ on $L^2(\Omega)$ norm but not on the $|\cdot|_1$ semi-norm. The term $\tilde{\phi}_\varepsilon$ is defined in a thin boundary layer that mostly force the approximation to satisfy the zero Dirichlet boundary condition.

Table IV compares the L^2 error between the proposed method and the multi-scale finite element presented in Reference [5]. We used $h_f=\frac{1}{3200}$ for $\varepsilon=\frac{1}{50}, \frac{1}{100}$, and $h_f=\frac{1}{1600}$ for $\varepsilon=\frac{1}{25}$. Observe that a factor 4 is obtained on our method for $\|u_\varepsilon^* - u_\varepsilon^{h,\hat{h},p}\|_0$ when u_ε^* is computed very accurately. We note that we do not obtain factor 4 from the $\varepsilon=\frac{1}{50}$, $h=32$ to $\varepsilon=\frac{1}{100}$, $h=64$ because h_f is not small enough to capture the fast scale. This is an explicit evidence that our method is more accurate than standard finite element methods on a very fine mesh.

In our numerical tests, we observed a very fast convergence of $v_e^{\hat{h},p}$ to the constant $\chi_e^{*,\hat{h},p}$ as $y_1\rightarrow -p$. Considering $p_1 < p_2 \in \{1,2\ldots 8\}$ we obtained that $\sup_{\{y_2\in[0,1],\, y_1\in[-p_2,-p_1]\}} |v_e^{\hat{h},p_1}(-p_1,y_2) - v_e^{\hat{h},p_2}(y_1,y_2)| \leqslant 10^{-14}$. This confirms that the numerical approximation for $\tilde{\phi}_\varepsilon^e$ given by Formula (10) is reasonable.

The figures below show the error evolution as we include the asymptotic expansion terms in our numerical approximation, for $h_f=\frac{1}{100}$, $h=\frac{1}{10}$, $\hat{h}=\frac{1}{50}$, $p=2$ and $\varepsilon=\frac{1}{20}$; Figure 1 is the

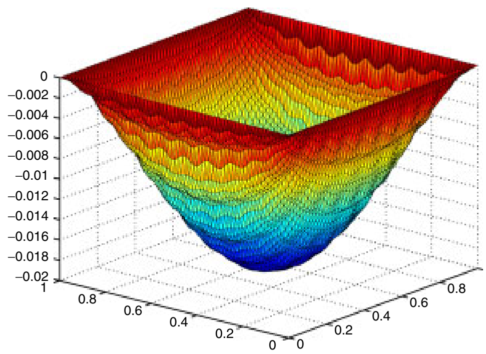

Figure 1. u_ε^*.

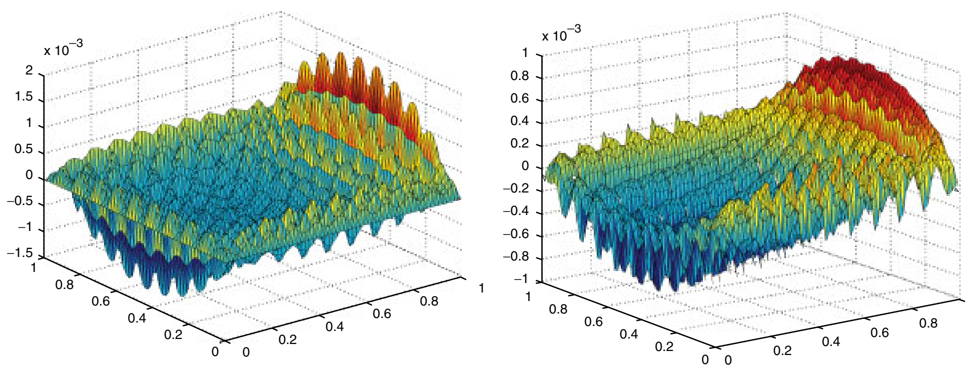

Figure 2. $u_\varepsilon^* - u_0^{h,\hat{h}}$ (left), and $u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}}$ (right).

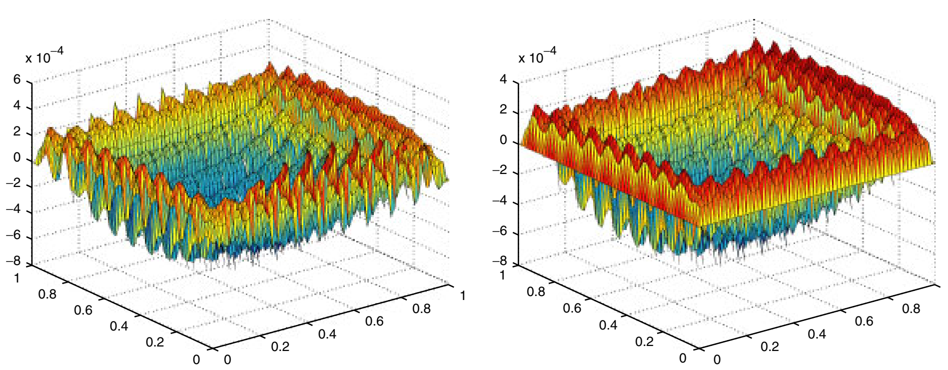

Figure 3. $u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}} - \varepsilon\bar{\phi}^{h,\hat{h},p}$ (left), and $u_\varepsilon^* - u_0^{h,\hat{h}} - \varepsilon u_1^{h,\hat{h}} - \varepsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}^{h,\hat{h},p})$ (right).

plot of the ‘exact’ solution u_ε^*. In Figure 2 from left to right we see that amplitude of error oscillations decreases when we include the approximation for u_1. It is possible to see an overall improvement in the error from Figure 2 (left) to Figure 3 (right) when the approximation for $\bar{\phi}$ is included, and finally in Figure 3 (left) we see that the zero boundary condition is satisfied when the complete approximation $u_0^{h,\hat{h}} + \varepsilon u_1^{h,\hat{h}} + \varepsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}^{h,\hat{h},p})$ is considered.

7. CONCLUSIONS

We propose a new method for approximating numerically the solution of Equation (1). This method is strongly based on the periodicity of the coefficients a_{ij}, and for this reason, it has relatively low computational cost with quasi-optimal error convergence rate. Although the convergence analysis presented in Reference [16] does not work for the quasi-periodic case $a_{ij}(x, x/\varepsilon)$, we believe that the numerical approximation presented here can be generalized for this case. This would be done by approximating matrix $a(x, x/\varepsilon)$ by $\sum_j a^j(x/\varepsilon) I_{K_j}(x)$, where I_{K_J} is the characteristic function for $K_j \in \mathscr{T}_k(\Omega)$, and then solving cell problem in each sub-domain K_j.

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REFERENCES

1. Hou T, Wu XH, Cai Z. Convergence of multi-scale finite element method for elliptic problems with rapidly oscillating coefficients. *Mathematics of Computation* 1999; **68**:913–943.
2. Babuska I. Solution of interface problem by homogenization I, II, III. *SIAM Journal on Mathematical Analysis* 1976; **7**:603–634 and 1977; **8**:923–937.
3. Bensoussan A, Lions JL, Papanicolaou G. *Asymptotic Analysis for Periodic Structures*. North-Holland: Amsterdam, 1980.
4. Efendiev YR, Hou T, Wu XH. Convergence of nonconforming multi-scale finite element method. *SIAM Journal on Numerical Analysis* 2000; **37**:888–910.
5. Hou T, Wu XH. A Multi-scale finite element method for elliptic problems in composite materials and porous media. *Journal of Computational Physics* 1997; **134**:169–189.
6. Allaire G, Brizzi R. A multiscale finite element method for numerical homogenization. *Internal Report*, CMAP, Ecole Polytechnique, 2004; 545.
7. Brezzi F. Interacting with the subgrid world. In *Proceedings of the Conference on Numerical Analysis*, Dundee, Scotland, 1999, Griffiths DF, Watson GA (eds). Chapman & Hall/CRC: New York/Boca Raton, FL, 2000; 69–82.
8. Arbogast T. Analysis of a two-scale, locally conservative subgrid upscaling for elliptic problems. *SIAM Journal on Numerical Analysis* 2004; **42**:576–598.
9. Sangalli G. Capturing small scales in elliptic problems using a residual-free bubbles finite element method. *SIAM Multi-scale Modelling Simulation* 2003; **1**:485–503.
10. Schwab C, Matache AM. *Generalized FEM for Homogenization Problems*. Lecture Notes in Computational Science and Engineering. Springer: Berlin, 2002.
11. E W., Engquist B. The heterogeneous multiscale method. *Communication in Mathematical Sciences* 2003; **1**:87–132.
12. Avellaneda M, Hou T, Papanicolaou G. Finite difference approximations for partial differential equations with rapidly oscillating coefficients. *Mathematical Modelling and Numerical Analysis* 1991; **25**:693–710.
13. Moscow S, Vogelius M. First-order corrections to the homogenized eigenvalues of a periodic composite medium. A convergence proof. *Proceedings of the Royal Society of Edinburgh* 1997; **127A**:1263–1299.
14. Oleinik O, Shamev AS, Yosifian GA. *Mathematical Problems in Elasticity and Homogenization*. North-Holland: Amsterdam, 1992.
15. Ladyzhenkaya OA, Ural'tseva NN. *Équations aux Dérivées Partielles de Type Elliptique*. Dunod: Paris, 1968.

16. Sarkis M, Versieux HM. Convergence analysis of numerical boundary correctors for elliptic equations with rapidly oscillating periodic coefficients, in press.
17. PETSc Web page. Balay S, Buschelman K, Gropp WD, Kaushik D, Knepley M, McInnes L, Smith B, Zhang H. http://www.mcs.anl.gov/petsc, 2001.
18. Arbogast T, Minkoff S, Keenan P. An operator-based approach to upscaling the pressure equation. In *Computation Methods in Water Resources XII*, Bentley L, Sykes J, Brebbia C, Gray W, Pinder G (eds). Balkema: Rotterdam, 2000; 161–166.
19. Braess D. *Finite Elements*, *Theory*, *Fast Solvers and Applications in Solid Mechanics*. Cambridge University Press: Cambridge, MA, 1977.
20. Brenner S, Scott R. *The Mathematical Theory of Finite Element Methods*. Springer: Berlin, 1994.

