Numer. Math. (1997) 77: 383–406

Numerische Mathematik

Electronic Edition

Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements

Marcus Sarkis[⋆]

Department of Computer Sciences, Colorado University, Boulder, CO 80309-0430, USA;
e-mail: msarkis@tigger.cs.colorado.edu

Received March 1, 1995 / Revised version received January 16, 1996

Summary. Two-level domain decomposition methods are developed for a simple nonconforming approximation of second order elliptic problems. A bound is established for the condition number of these iterative methods, that grows only logarithmically with the number of degrees of freedom in each subregion. This bound holds for two and three dimensions and is independent of jumps in the value of the coefficients and number of subregions. We introduce face coarse spaces, and isomorphisms to map between conforming and nonconforming spaces.

Mathematics Subject Classification (1991): 65F10, 65N30, 65N55

1. Introduction

The purpose of this paper is to develop domain decomposition methods for second order elliptic partial differential equations approximated by a simple nonconforming finite element method, the nonconforming P_1 elements. We first consider a variant of a two-level additive Schwarz method introduced in 1987 by Dryja and Widlund [15] for a conforming case. In this method, a preconditioner is constructed from the restriction of the given elliptic problem to overlapping subregions into which the given region has been decomposed. In addition, in order to enhance the convergence rate, the preconditioner includes a coarse problem of lower dimension. The construction of this component is the most interesting part

⋆ This work has been supported in part by a Brazilian graduate student fellowship from Conselho Nacional de Desenvolvimento Cientifico e Tecnologico - CNPq, in part by a Dean's Dissertation New York University Fellowship, in part by the National Science Foundation under Grant NSF-CCR-9204255 and the U.S. Department of Energy under contract DE-FG02-92ER25127, and in part by the NSF grant ASC-9406582 and the NSF grant ASC-9217394. This work was mainly done while the author was a student in Courant Institute of Mathematical Sciences

of this work; we introduce *nonstandard coarses spaces* for problems with discontinuous coefficients, and show that the condition number of the corresponding iterative methods is bounded by $C\,(1+\log(H/h))$. Here, H and h are the mesh sizes of the global and local problems, respectively. We also note that this bound is independent of the variations of the coefficients across the subregion interfaces.

Our motivation for considering the P_1-nonconforming space comes from the equivalence established by Arnold and Brezzi [1] between nonconforming methods and Raviart-Thomas mixed methods. The latter methods are useful in applications for which an accurate approximation to the flux variable of the elliptic problem is required and the solution (of the elliptic problem) is not sufficiently smooth; see Sarkis [26].

This paper is an extended version of a technical report completed in March 1993 [24]. That work had been inspired by earlier multilevel studies, cf. Brenner [4], Oswald [23] as well as by recent work by Dryja, Smith, and Widlund [14], but a number of additional technical difficulties had to be overcome. They are primarily related to our efforts to treat quite general coefficients. One of the main ideas relates to the construction of certain *isomorphisms*, or *local interpolators*, which map between conforming and nonconforming spaces and to obtaining several results for the nonconforming case which are known for the conforming case.

In this work, we also introduce the *face based* and the β*-Neumann-Neumann* coarse spaces, that have the following characteristics. The values at the nonconforming nodes are constant on each edge (face) of the subregions and the values at the other nonconforming nodes are given by a piecewise discrete nonconforming P_1 harmonic extension, or by a simple but nonstandard interpolation formula, an (*approximate harmonic extension*). In the latter case, and for triangular (tetrahedral) substructures, the value at any nonconforming node in the interior of a subregion is a convex combination of the three (four) values given on the boundary.

We believe that our isomorphisms were first introduced in a short version of [24] that was entered into Copper Mountain student competition in mid-December 1992. We note that recently considerable attention has been focused on related techniques for domain decomposition methods with nonconforming spaces; for second order scalar problems (cf. Brenner [3], Cowsar [9, 10], Cowsar, Mandel, and Wheeler [11], Sarkis [25, 26]), for plate elements (cf. Brenner [2]), and for non-nested meshes (cf. Cai [5, 6], Chan, Smith, and Zou [7]). We also note the *face based* coarse spaces and *approximate harmonic extensions*, which were originally introduced in [24], also have been discussed in the conforming case in recent work by Dryja, Sarkis, and Widlund [13], and Dryja, Smith, and Widlund [14].

2. Differential and finite element model problems

To simplify the presentation, we assume that Ω is an open, bounded, polygonal region of diameter 1 in the plane, with boundary $\partial\Omega$. In a separate section, we extend all our results to the three dimensional case.

We introduce a partition of Ω as follows. In a first step, we divide the region Ω into nonoverlapping triangular substructures $\Omega_i, i = 1, \cdots, N$. Adopting common assumptions in finite element theory, cf. Ciarlet [8], all substructures are assumed shape regular, quasi-uniform and to have no dead points, i.e. each interior edge is the intersection of the boundaries of two triangular regions. We can show that the theory also holds if we choose nontriangular substructures, where the boundary of each substructure is a composition of several polygonal edges, and each of them is the intersection of two substructures. Naturally, we need assumptions related to the nondegeneracy of this partition. Initially, we restrict our exposition to the case of triangular substructures since the main ideas can be seen in this case. This partition induces a coarse mesh $\mathscr{T}^H$ and we also introduce a mesh parameter $H = \max\{H_1, \cdots, H_N\}$ where H_i is the diameter of Ω_i. Later, we will extend the results to nontriangular substructures. The quasiuniformity assumption is not required for the coarse triangulation $\mathscr{T}^H$ since all arguments are local.

We study the following selfadjoint second order elliptic problem:
Find $u \in H_0^1(\Omega)$, such that

$$a(u, v) = f(v) \quad \forall\, v \in H_0^1(\Omega)\ , \tag{1}$$

where

$$a(u, v) = \sum_{i=1}^{N} \int_{\Omega_i} \rho_i\, \nabla u \cdot \nabla v\, dx \quad \text{and} \quad f(v) = \int_{\Omega} f v\, dx \quad \text{for} f \in L^2\ .$$

Here, the ρ_i are constants values such that $\rho_i \geq \lambda_{\min} > 0 \quad \forall i$. This includes cases where there is a great variation in the value of the ρ_i across substructures interfaces. We remark that there is no difficulty in extending the analysis and the results to the case where each ρ_i varies mildly inside each substructure Ω_i.

In a second step, we obtain the elements τ_j^h in the finest triangulation by subdividing the substructures into triangles in such a way that they are shape regular, and quasi-uniform. The mesh parameter h is the diameter of the smallest element and denote this triangulation by $\mathscr{T}^h$. Similarly, we assume the triangulation $\mathscr{T}^h$ has no dead points.

Let $\mathscr{E}_{ij}$ represents an open *edge* of a substructure Ω_i; sometimes we use $\mathscr{E}_{ij}$ to represent an open edge shared by two substructures Ω_i and Ω_j. We denote the interface between the subdomains by $\Gamma = \cup\partial\Omega_i \backslash \partial\Omega$. We denote by *CR nodal points* the nonconforming nodal points, i.e. the midpoints of the edges of elements in $\mathscr{T}^h$. The sets of CR nodal points belonging to $\bar{\Omega}$, $\bar{\Omega}_i$, $\partial\Omega$, $\mathscr{E}_{ij}$, $\partial\Omega_i$, and Γ are denoted by $\bar{\Omega}_h^{\mathrm{CR}}$, $\bar{\Omega}_{i,h}^{\mathrm{CR}}$, $\partial\Omega_h^{\mathrm{CR}}$, $\mathscr{E}_{ij,h}^{\mathrm{CR}}$, $\partial\Omega_{i,h}^{\mathrm{CR}}$, and Γ_h^{CR}, respectively.

Definition 1. The nonconforming P_1 element spaces (cf. Crouzeix and Raviart [12] on the h-mesh and H-mesh are given by

$\tilde{V}^h(\Omega) = \{v \mid v$ linear in each triangle $\tau_j^h \in \mathscr{T}^h$,
v continuous at the CR nodes of $\Omega_h^{\mathrm{CR}}\}$,
$\tilde{V}_0^h(\Omega) = \{v \mid v \in \tilde{V}^h(\Omega)$ and $v = 0$ at the CR nodes of $\partial\Omega_h^{\mathrm{CR}}\}$,
$\tilde{V}^H(\Omega) = \{v \mid v$ linear in each triangle in $\mathscr{T}^H$,
v continuous at the midpoints of the edges of $\mathscr{T}^H\}$, and
$\tilde{V}_0^H(\Omega) = \{v \mid v \in \tilde{V}^H(\Omega)$ and $v = 0$ at the
midpoints of edges of $\mathscr{T}^H$ that belong to $\partial\Omega\}$.

These spaces are nonconforming and non-nested since $\tilde{V}_0^H(\Omega) \not\subset \tilde{V}_0^h(\Omega) \not\subset H_0^1(\Omega)$.

Let Σ be a region contained in Ω such that $\partial\Sigma$ does not cut through any element. Denote by $\tilde{V}^h(\Sigma)$ and $\mathscr{T}^h(\bar{\Sigma})$ the space $\tilde{V}^h$ and the triangulation $\mathscr{T}^h$ restricted to $\bar{\Sigma}$, respectively.

Given $u \in \tilde{V}^h(\Sigma)$, we define a discrete weighted energy seminorm by:

$$|u|^2_{H^1_{\rho,h}(\Sigma)} = a^h_\Sigma(u,u)\,, \tag{2}$$

where

$$a^h_\Sigma(u,v) = \sum_{T \in \mathscr{T}^h_{\mid \bar{\Sigma}}} \int_T \rho(x)\ \nabla u \cdot \nabla u\ dx. \tag{3}$$

In a similar fashion, we define the inner product $a^H_\Omega(u,v)$ and the seminorm $|u|_{H^1_{a,H}(\Sigma)}$ for $u, v \in \tilde{V}^H(\Sigma)$. In order to avoid unnecessary notations, we drop the subscript Σ when the integration is over Ω and the subscript ρ when $\rho = 1$.

The discrete problem associated with (1) is given by:
Find $u \in \tilde{V}_0^h$, such that

$$a^h(u,v) = f(v) \quad \forall\ v \in \tilde{V}_0^h(\Omega). \tag{4}$$

We note that $|\cdot|_{H^1_{\rho,h}(\Omega)}$ is a norm for the space $\tilde{V}_0^h$, because if $|u|_{H^1_{\rho,h}(\Omega)} = 0$, then u is constant in each element. By the continuity at the CR nodes and the zero boundary condition, we obtain $u = 0$. Therefore, we obtain uniqueness and existence of the solution for (4).

We obtain similar stability results as for the conforming case.

We also define the weighted L^2 norm by:

$$\|u\|^2_{L^2_\rho(\Sigma)} = \int_\Sigma \rho(x)\ |u(x)|^2\ dx \quad \text{for } u \in (\tilde{V}^h(\Sigma) + \tilde{V}^H(\Sigma) + L^2(\Sigma)), \tag{5}$$

and a weighted energy norm by:

$$\|u\|^2_{H^1_{\rho,h}(\Sigma)} = |u|^2_{H^1_{\rho,h}(\Sigma)} + \frac{1}{(d\,(\Sigma))^2}\|u\|^2_{L^2_\rho(\Sigma)}, \tag{6}$$

where $d\,(\Sigma)$ is the diameter of the region Σ.

We introduce the following notations: $u \preceq v$, $w \succeq x$, and $y \asymp z$ meaning that there are positive constants C and c such that

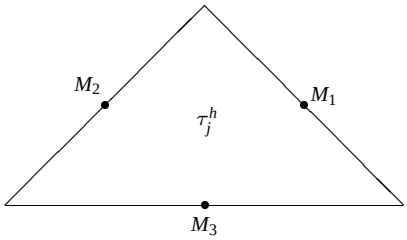

Fig. 1.

$$u \leq C\, v, \qquad w \geq c\, x \quad \text{and} \quad c\, z \leq y \leq C\, z, \quad \text{respectively.}$$

Here C and c are independent of the variables appearing in the inequalities and the parameters related to meshes, spaces and, especially, the weight ρ. Sometimes, we will use $\leq$ to stress that $C = 1$.

Sometimes is more convenient to evaluate a norm of a finite element function in terms of the values of this function at the CR nodal points. By first working on a reference element and then using the assumption that the elements are shape regular, we obtain the following lemma:

Lemma 1. *For* $u \in \tilde{V}^h(\Sigma)$,

$$\|u\|_{L^2_\rho(\Sigma)} \asymp h^2 \sum_{\tau_j^h \in \mathcal{T}^h(\Sigma)} \rho(\tau_j^h)\,(u^2(M_1) + u^2(M_2) + u^2(M_3)), \tag{7}$$

and

$$|u|^2_{H^1_{\rho,h}(\Sigma)} \asymp \sum_{\tau_j^h \in \mathcal{T}^h(\Sigma)} \rho(\tau_j^h)\,\{(u(M_1) - u(M_2))^2 + (u(M_2) - u(M_3))^2 + (u(M_3) - u(M_1))^2\}. \tag{8}$$

where M_1, M_2, M_3 *are the CR nodes of the triangle* τ_j^h *as in Fig. 1.*

We stress the the formulas (7) and (8) are for the two-dimensional case. There are three-dimensional version of these formulas; the main difference is that we have to multiply the right hand sides by an additional factor h.

An inverse inequality can be obtained by using only local properties. It is easy to see that for $u \in \tilde{V}^h(\Sigma)$,

$$|u|_{H^1_{\rho,h}(\Sigma)} \preceq h^{-1}\|u\|_{L^2_\rho(\Sigma)}. \tag{9}$$

We note that (9) also holds for the three-dimensional case.

3. Additive Schwarz schemes

We now describe the special additive Schwarz method introduced by Dryja and Widlund; see e.g. [16, 19]. In this method, we cover Ω by overlapping subregions obtained by extending each substructure Ω_i to a larger region Ω_i'. We assume that the overlap is δ_i, where δ_i is the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$, and we denote by δ the minimum of the δ_i. We also assume that $\partial\Omega_i'$ does not cut through any element. We make the same construction for the substructures that meet the boundary except that we cut off the part of Ω_i' that is outside of Ω.

For each Ω_i', a nonconforming P_1 finite element triangulation is inherited from $\mathscr{T}^h(\Omega)$. The corresponding finite element space is defined by

$$\tilde{V}_i = \{v \mid v \in \tilde{V}_0^h, \text{ support of } v \subset \bar{\Omega}_i'\}, \qquad i = 1, \cdots, N. \tag{10}$$

A coarse space $\tilde{V}_0 \subset \tilde{V}_0^h(\Omega)$ is given as the range of a certain composition of interpolation and prolongation operators. The most fundamental and original coarse space for nonconforming P_1 spaces is the *face based* coarse space with an *approximate discrete harmonic extension*. This space, which we denote by $\tilde{V}_0^{\tilde{F}}$, can also be defined as the range of an operator $\tilde{I}_H^h$. The *prolongation operator* $\tilde{I}_H^h$ will be defined later.

Our finite element space is represented as a sum of $N+1$ subspaces

$$\tilde{V}_0^h = \tilde{V}_0 + \tilde{V}_1 + \cdots + \tilde{V}_N. \tag{11}$$

We introduce projection operators $\tilde{P}_i : \tilde{V}_0^h \to \tilde{V}_i$, $i = 0, \cdots, N$, by

$$a^h(\tilde{P}_i w, v) = a^h(w, v) \quad \forall\, v \in \tilde{V}_i, \tag{12}$$

and the operator $\tilde{P} : \tilde{V}_0^h \to \tilde{V}_0^h$, by

$$\tilde{P} = \tilde{P}_0 + \tilde{P}_1 + \cdots + \tilde{P}_N. \tag{13}$$

In a case in which $\tilde{V}_0 = \tilde{V}_0^{\tilde{F}}$, the matrix form of $\tilde{P}_0$ is given by

$$\tilde{P}_0 = \tilde{I}_H^h (\tilde{I}_H^{h\,\mathrm{T}} K \tilde{I}_H^h)^{-1} \tilde{I}_H^{h\,\mathrm{T}} K, \tag{14}$$

where K is the finite element operator associated with $a_h(\cdot,\cdot)$.

We replace the problem (4) by

$$\tilde{P}u = g, \quad g = \sum_{i=0}^{N} g_i \quad \text{where} \quad g_i = \tilde{P}_i u. \tag{15}$$

By construction, (4) and (15) have the same solution. We point out that g_i can be computed, without knowledge of u, since we can find g_i by solving

$$a^h(g_i, v) = a^h(u, v) = f(v) \quad \forall v \in V_i^h. \tag{16}$$

It is very easy to check that the operator $\tilde{P}$ is symmetric with respect to the inner product $a^h(\cdot,\cdot)$. The reason for replacing the problem (4) by (15) is that, we

can transform a large ill-conditioned system into a very well conditioned system problem at the expense of solving many small ill-conditioned and independent linear problems. The equation (15) is typically solved by a conjugate gradient method, without further preconditioning, using $a^h(\cdot,\cdot)$ as the inner product. In order to see that the problem (15) has u as the unique solution, and also to estimate the rate of convergence of the preconditioned conjugate gradient method, we need to obtain upper and lower bounds for the spectrum of $\tilde{P}$. A lower bound is obtained by using the following lemma; cf. Dryja and Widlund [16–18], Matsokin and Nepomnyaschikh [21], and Zhang [27].

Lemma 2. *Let $\tilde{P}_i$ be the operators defined in equation (12) and let $\tilde{P}$ be given by (13). Then*

$$a^h(\tilde{P}^{-1}v, v) = \min_{v=\sum v_i} \sum a^h(v_i, v_i),\ v_i \in \tilde{V}_i^h. \tag{17}$$

Therefore, if a representation $v = \sum v_i$ can be found such that

$$\sum_{i=0}^{N} a^h(v_i, v_i) \le C_0^2 a^h(v, v), \quad \forall v \in \tilde{V}_0^h, \tag{18}$$

then

$$\lambda_{\min}(\tilde{P}) \ge C_0^{-2}.$$

4. Isomorphism

We first introduce a *local equivalence map* (*isomorphism*) in order to obtain some inequalities and local properties of our nonconforming space. With the help of that map, we can extend some results that are known for the piecewise linear conforming elements to our nonconforming case. We point out that all the isomorphisms introduced in this paper are carefully defined so that our analysis also holds for any triangulation that is shape regular and nonuniform.

Let $V^{\frac{h}{2}}(\Omega_i)$ be the conforming space of continuous, piecewise linear functions on the triangulation $\mathscr{T}^{\frac{h}{2}}(\Omega_i)$, where the $h/2$-mesh is obtained by joining midpoints of the edges of elements of $\mathscr{T}^h(\Omega_i)$. The set of vertices in $\mathscr{T}^h$ belonging to $\bar{\Omega}_i$, and $\partial\Omega_i$ are denoted by $\bar{\Omega}_{i,h}$, and $\partial\Omega_{i,h}$, respectively.

We define the *local equivalence map* $\mathscr{M}_i^{\mathscr{E}_{ij}} : \tilde{V}^h(\Omega_i) \to V^{\frac{h}{2}}(\Omega_i)$, as follows:

Isomorphism 1. Given $u \in \tilde{V}^h(\Omega_i)$, and an edge $\mathscr{E}_{ij}$ of the substructure Ω_i, define $\mathscr{M}_i^{\mathscr{E}_{ij}} u \in V^{\frac{h}{2}}(\Omega_i)$ by the values of $\mathscr{M}_i^{\mathscr{E}_{ij}} u$ at the four sets of points (cf. Fig. 2):

i) If $P \in \bar{\Omega}_{i,h}^{\mathrm{CR}}$, then

$$\mathscr{M}_i^{\mathscr{E}_{ij}} u(P) = u(P).$$

ii) If $P \in \bar{\Omega}_{i,h} \backslash \partial\Omega_{i,h}$, and the τ_j^h are the elements that have P as a vertex, then

$$\mathscr{M}_i^{\mathscr{E}_{ij}} u(P) = \text{mean of} \quad u|_{\tau_j^h}(P).$$

Here $u|_{\tau_j^h}(P)$, is the limit value of $u(x)$ when $x \in \tau_j^h$ approaches P.

iii) If $\mathscr{V}$ is a vertex of the substructure Ω_i and an end point of $\mathscr{E}_{ij}$, and $\mathscr{V}_{\mathscr{E}_{ij}}$ the CR node on $\mathscr{E}_{ij,h}^{\mathrm{CR}}$ that is the next neighbor of $\mathscr{V}$, then

$$\mathscr{M}_i^{\mathscr{E}_{ij}} u(\mathscr{V}) = u(\mathscr{V}_{\mathscr{E}_{ij}}).$$

iv) If $Q \in \partial\Omega_{i,h}$ and is not in the case iii), and Q_l and Q_r the two CR nodes of $\partial\bar{\Omega}_{i,h}^{\mathrm{CR}}$ that are the next neighbors of Q, then

$$\mathscr{M}_i^{\mathscr{E}_{ij}} u(Q) = \frac{|Q_l Q|}{|Q_l Q| + |Q Q_r|} u(Q_l) + \frac{|Q Q_r|}{|Q_l Q| + |Q Q_r|} u(Q_r).$$

Here $|Q_l Q|$ is the length of the segment $Q_l Q$.

In a case where the triangulation $\mathscr{T}^h(\Omega_i)$ is uniform, e.g. as in Fig. 2, Case ii) becomes

$$\mathscr{M}_i^{\mathscr{E}_{ij}} u(P) = \frac{1}{6} \sum_{i=j}^{6} u|_{\tau_j^h}(P).$$

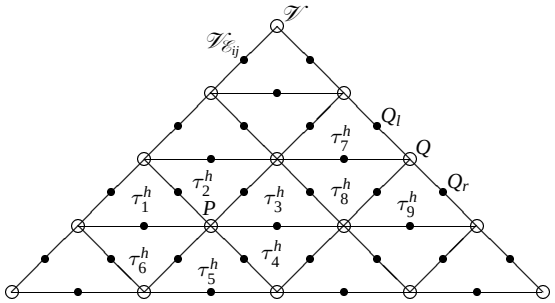

Fig. 2.

We can easily prove (cf. Sarkis [26]):

Lemma 3. *Given $u \in \tilde{V}^h(\Omega_i)$. Then,*

$$|\mathscr{M}_i^{\mathscr{E}_{ij}} u|_{H^1_{\rho,h}(\Omega_i)} \asymp |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{19}$$

$$\|\mathscr{M}_i^{\mathscr{E}_{ij}} u\|_{L^2_\rho(\Omega_i)} \asymp \|u\|_{L^2_\rho(\Omega_i)}, \tag{20}$$

and

$$\int_{\mathscr{E}_{ij}} \mathscr{M}_i^{\mathscr{E}_{ij}} u(s)\, ds = \int_{\mathscr{E}_{ij}} u(s)\, ds. \tag{21}$$

Remark 1. The Isomorphism 1 is a local construction. Therefore, Lemma 3 holds locally in each Ω_i but not simultaneously for every i. In fact, we can prove that it is impossible to construct a global Isomorphism $\mathscr{M} : \tilde{V}^h(\Omega) \to V^{\frac{h}{2}}(\Omega)$ such that (19) and (20) hold for every i simultaneously. The problem is how to define the values at the vertices of the substructures that are cross points. We note, however, that for problems with constant or quasi-monotone coefficients, discussed in [13], a global isomorphism $\mathscr{M}$ can be defined in such way that (19) and (20) hold simultaneously for every Ω_i.

5. Interpolation operator

In this section, we introduce an interpolation operator for P_1-nonconforming spaces and establish that it is stable. This operator plays an important role in the analysis of our algorithms.

Definition 2. Let $v \in \tilde{V}^h(\Omega)$. The Interpolation operator $I_h^H : \tilde{V}^h(\Omega) \to \tilde{V}^H(\Omega)$, is given by:

i) If P_{ij} is the midpoint of the edge $\mathscr{E}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, then

$$(I_h^H v)(P_{ij}) = \frac{1}{|\mathscr{E}_{ij}|} \int_{\mathscr{E}_{ij}} v|_{\bar{\Omega}_i}(x)\, dx = \frac{1}{|\mathscr{E}_{ij}|} \int_{\mathscr{E}_{ij}} v|_{\bar{\Omega}_j}(x)\, dx.$$

ii) If P_{ij} is the midpoint of an edge $\mathscr{E}_{ij}$ common to $\bar{\Omega}_i$ and $\partial\Omega$, then

$$(I_h^H v)(P_{ij}) = \frac{1}{|\mathscr{E}_{ij}|} \int_{\mathscr{E}_{ij}} v|_{\bar{\Omega}_i}(x)\, dx.$$

Here, $v|_{\bar{\Omega}_i}(x)$ is the limit value of $v(y)$ when $y \in \Omega_i$ approaches x. We note that the second equality in part i) of Definition 2 follows from the fact that the mean of v on each edge of an element $\tau_j^h \in \mathscr{T}^h(\Omega)$ is equal to $v(M_1)$, where M_1 is the CR node of this edge. It is important to note that the value of $(I_h^H v)(P_{ij})$ depends only on the values of v on the interface $\mathscr{E}_{ij}$. This allows us to obtain stability properties that are independent of the differences of $\rho(x)$ across the substructure interfaces. It is easy to see from Definition 2 that

$$(I_h^H \tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^H(\Omega).$$

Before studying the stability properties of this operator, we need two lemmas for the piecewise linear conforming finite element space.

The following lemma is a Poincaré-Friedrichs inequality. The idea of the proof can be found in Ciarlet (Theorem 6.1) [8] and in Nečas (Chapter 2.7.2) [22].

Lemma 4. *Let γ be a subset of $\partial\Omega_i$, such that γ and $\partial\Omega_i$ have measures on the order of H. Then,*

$$\|\overline{u}\|^2_{L^2(\Omega_i)} \preceq H^2|\overline{u}|^2_{H^1(\Omega_i)} + (\int_\gamma \overline{u}(x)\,dx)^2, \quad \forall \overline{u} \in H^1(\Omega_i). \tag{22}$$

As a consequence, if $\int_\gamma \overline{u}(x)\,dx = 0$, *we have the Poincaré inequality*

$$\|\overline{u}\|_{L^2_\rho(\Omega_i)} \preceq H\,|\overline{u}|_{H^1_{\rho,h}(\Omega_i)}. \tag{23}$$

The next lemma is a Poincaré-Friedrichs inequality for nonconforming P_1 elements. It is obtained by using Lemmas 3, and 4.

Lemma 5. *Let* $u \in \tilde{V}^h(\Omega_i)$, *where* Ω_i *is a triangular substructure of diameter O(H). Let* γ *be* $\partial\Omega_i$ *(or an edge of* $\partial\Omega_i$*). Then,*

$$\|u\|^2_{L^2(\Omega_i)} \preceq H^2|u|^2_{H^1_h(\Omega_i)} + (\int_\gamma u(s)\,ds)^2. \tag{24}$$

As a consequence, if $\int_\gamma u(s)\,ds = 0$, *we have the Poincaré inequality*

$$\|u\|_{L^2_\rho(\Omega_i)} \preceq H\,|u|_{H^1_{\rho,h}(\Omega_i)}. \tag{25}$$

The next lemma gives an example of an operator locally $L^2_\rho-$ and $H^1_{\rho,h}-$stable.

Lemma 6. *Let* $\overline{u} \in H^1(\Omega_i)$, *where* Ω_i *is a triangular substructures of diameter of O(H). Define a linear function* $\overline{u}_H$ *in* Ω_i *by*

$$\overline{u}_H(P_{ij}) = \frac{1}{|\mathscr{E}_{ij}|}\int_{\mathscr{E}_{ij}} \overline{u}(s)\,ds \quad j = 1,2,3, \tag{26}$$

where the $\mathscr{E}_{ij}$ *are the edges of* Ω_i, *and* P_{ij} *is the midpoint of* $\mathscr{E}_{ij}$. *Then*

$$|\overline{u}_H(P_{ij})|^2 \preceq |\overline{u}|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|\overline{u}\|^2_{L^2(\Omega_i)}. \tag{27}$$

$$|\overline{u}_H|_{H^1_{\rho,h}(\Omega_i)} \preceq |\overline{u}|_{H^1_{\rho,h}(\Omega_i)}, \tag{28}$$

and

$$\|\overline{u}_H - \overline{u}\|_{L^2_\rho(\Omega_i)} \preceq H\,|\overline{u}|_{H^1_{\rho,h}(\Omega_i)}. \tag{29}$$

Proof. Consider initially a subregion Ω_i with diameter of 1. Using that $|\mathscr{E}_{ij}| =$ O(1), the Cauchy-Schwarz inequality and a trace theorem, we have

$$|\overline{u}_H(P_{ij})|^2 \asymp |\int_{\mathscr{E}_{ij}} \overline{u}(x)\,dx|^2 \preceq \|\overline{u}\|^2_{L^2(\mathscr{E}_{ij})}$$

$$\preceq \|\overline{u}\|^2_{H^{\frac{1}{2}}(\mathscr{E}_{ij})} \preceq \|\overline{u}\|^2_{H^1(\mathscr{E}_{ij})} \preceq \|\overline{u}\|^2_{L^2(\mathscr{E}_{ij})} + |\overline{u}|^2_{H^1(\mathscr{E}_{ij})}.$$

We obtain (27) by returning to a region of diameter H.

Note that using the first part of this lemma, for any constant c

$$|\overline{u}_H|^2_{H^1(\Omega_i)} \tag{30}$$

$$\asymp |\overline{u}_H(P_{i1}) - \overline{u}_H(P_{i2})|^2 + |\overline{u}_H(P_{i2}) - \overline{u}_H(P_{i3})|^2 + |\overline{u}_H(P_{i3}) - \overline{u}_H(P_{i1})|^2$$

$$\preceq \|\bar{u} - c\|^2_{H^1(\Omega_i)}.$$

By choosing $c = \bar{u}(P_{i1})$ and $\gamma = \mathscr{E}_{i1}$, we can apply Lemma 4 and obtain the $H^1_{\rho,h}(\Omega)$-stability (28).

We now prove the $L^2_\rho(\Omega)$ error stability. Since $\bar{u} - \bar{u}_H$ has mean zero on $\partial\Omega_i$, we can apply the Poincaré inequality (25) and obtain

$$\|\bar{u} - \bar{u}_H\|_{L^2(\Omega_i)} \preceq H\, |\bar{u} - \bar{u}_H|_{H^1(\Omega_i)}. \tag{31}$$

Using the second part of this lemma, we obtain the $L^2_\rho(\Omega)$ error stability (29). □

The next lemma shows that the interpolation operator I^H_h, introduced in Definition 2, is locally $L^2_\rho-$ and $H^1_{\rho,h}-$stable.

Lemma 7. *Let $u \in \tilde{V}^h(\Omega)$. Then $u_H = I^H_h u \in \tilde{V}^H(\Omega)$ satisfies the following properties*

$$(I^H_h \tilde{V}^h_0(\Omega)) \subset \tilde{V}^H_0(\Omega), \tag{32}$$

$$|u_H|_{H^1_{\rho,H}(\Omega_i)} \preceq |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{33}$$

and

$$\|u_H - u\|_{L^2_\rho(\Omega_i)} \preceq H\, |u|_{H^1_{\rho,h}(\Omega_i)}, \quad i = 1, \cdots, N. \tag{34}$$

Proof. Let $u_H = I^H_h u \in \tilde{V}^H(\Omega)$ and $u_{i1} = \mathscr{M}_i^{\mathscr{E}_{i1}} u \in V^{\frac{h}{2}}(\Omega_i)$, and let $\bar{u}_H(P_{i1}) \in H^1(\Omega_i)$ be given by (26). Using (21) and Definition 2, we have

$$u_H(P_{i1}) = \bar{u}_H(P_{i1}). \tag{35}$$

Therefore, from (35), (27), and Lemma 3, we have

$$|u_H(P_{i1})|^2 = |\bar{u}_H(P_{i1})|^2 \preceq |\bar{u}|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|\bar{u}\|^2_{L^2(\Omega_i)} \tag{36}$$

$$\preceq |u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)}.$$

We also obtain the same estimate for $|u_H(P_{i2})|$ and $|u_H(P_{i3})|$.

The rest of the proof is similar to that of Lemma 6. We now use Lemma 5, the Poincaré inequality for nonconforming elements. □

6. The face based basis

In this section, we introduce prolongation operators and establish that they are stable. The range of each of these operators will serve as a coarse space in our algorithms.

Definition 3. The Prolongation Operator, $I^h_H : \tilde{V}^H \to \tilde{V}^h$, is given by:

i) Let $\mathscr{E}_{ij}, j = 1, 2, 3$, be the edges of a substructures Ω_i. For all CR nodes $q \in \mathscr{E}^{\mathrm{CR}}_{ij,h}$, $j = 1, 2, 3$, and $i = 1, \cdots, N$, let $(I^h_H u_H)(q) = u_H(P_{ij})$, where P_{ij} is the midpoint of the edge $\mathscr{E}_{ij}$.

ii) Given $I_H^h u_H$ at the CR nodal points of $\Gamma = \cup_i \partial\Omega_i$ from i), let $I_H^h u_H(\Omega)$ be the nonconforming P_1 harmonic extension inside each Ω_i.

It is easy to check that $u_h = I_H^h u_H \in \tilde{V}^h(\Omega)$, and $(I_H^h \tilde{V}_0^H(\Omega)) \subset \tilde{V}_0^h(\Omega)$. A disadvantage of step ii) is that we have to solve exactly a local Dirichlet problem for each substructure in order to obtain the harmonic extension. Other extensions can be used, which we call *approximate harmonic extensions*. They are given by simple explicit formulas with the same L^2_ρ and $H^1_{\rho,h}$ stability properties as the harmonic one.

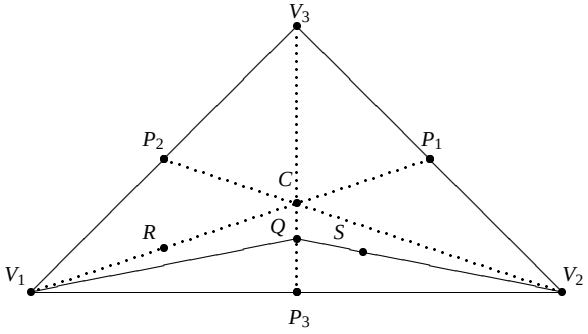

Fig. 3.

Let P_j, $j = 1, 2, 3$, be the midpoints of the edges of Ω_i, and let V_j be the vertex of Ω_i that is opposite to P_j. Let C be the barycenter of the triangle Ω_i, i.e. the intersection of the line segments connecting V_j to P_j.

Extension 1. The construction of an approximate harmonic extension in Ω_i is defined by the following steps (see Fig. 3):

i) Let
$$\bar{u}(C) = \frac{1}{3}\{u_H(P_1) + u_H(P_2) + u_H(P_3)\}.$$

ii) For a point R that belongs to a line segment that connects C to a vertex V_j, let
$$\bar{u}(R) = \bar{u}(C).$$

iii) For a point Q that belongs to a line segments connecting C to P_k, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(P_k)$, i.e by
$$\bar{u}(Q) = \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(P_k).$$
Here $\lambda(Q) = \text{dist}(Q, P_k)/\text{dist}(C, P_k)$.

iv) For a point S that belongs to the line segment connecting the previous point Q to a vertex V_ℓ, with $\ell \neq k$, let
$$\bar{u}(S) = \bar{u}(Q).$$

v) Finally, let $\tilde{I}_H^h u_H(\Omega_i) = I_{i,h}^{\mathrm{CR}} \bar{u}$, where $I_{i,h}^{\mathrm{CR}}$ is the interpolation operator into the space $\tilde{V}^h(\Omega_i)$ that preserves the values of a function at the CR nodes $\bar{\Omega}_{i,h}^{\mathrm{CR}}$.

Note that the function $\bar{u}$ just constructed is continuous except at the vertices V_j of Ω_i. We note that several other extensions can be easily constructed; Sarkis [26]. The construction given above is a generalization of the partition of unity introduced by Dryja and Widlund in [16]. We note Dryja and Widlund [16] use this partition only as a tool for analyzing their algorithms, while here we use the approximate harmonic extension also to define our algorithms.

The next lemma shows that the extension given above have quasi-optimal energy stability.

Lemma 8. *Let $u_H \in \tilde{V}^H(\Omega)$. Then*

$$|\tilde{I}_H^h u_H|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + \log(H/h))^{\frac{1}{2}} \, |u_H|_{H^1_{\rho,H}(\Omega_i)}, \tag{37}$$

and

$$\|\tilde{I}_H^h u_H - u_H\|_{L^2_\rho(\Omega_i)} \preceq H \, |u_H|_{H^1_{\rho,H}(\Omega_i)} \quad i = 1, \cdots, N. \tag{38}$$

Proof. Let $\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}} \in \tilde{V}^h(\Omega_i), j = 1, 2, 3$, be the approximate harmonic extension defined by Extension 1, constructed from the boundary values $\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}} = 1$ at the CR nodes $\mathscr{E}_{ij,h}^{\mathrm{CR}}$, and $\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}} = 0$ at the other CR nodes of $\partial\Omega_{i,h}^{\mathrm{CR}}$. It is easy to see that the $\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}}$ form a basis of all approximate harmonic extensions that take constant values on the edges of the substructure. Using ideas of Dryja and Widlund [16], it is easy to show that if a point x belongs to the interior of an element of Ω_i, then $|\nabla \vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}}(x)|$ is bounded by C/r, where r is the minimum distance from x to any vertex of Ω_i. Note that any element that touches a vertex of Ω_i provides an order one contribution to the energy seminorm. To estimate the contribution to the energy seminorm from the rest of the substructure, we introduce polar coordinate systems centered at the vertices of Ω_i. Then,

$$|\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}}|^2_{H^1_h(\Omega_i)} \preceq 1 + \int \int_h^H r^{-2} \, r \, dr \, d\varphi \preceq 1 + \log(H/h). \tag{39}$$

Since the partition of unity $\vartheta_{\mathscr{E}_{ij}}^{\mathrm{CR}}$ forms a basis, it is easy to see that

$$|\tilde{I}_H^h u_H|^2_{H^1_h(\Omega_i)} \preceq (1 + \log(H/h)) \{|u_H(P_1)|^2 + |u_H(P_2)|^2 + |u_H(P_3)|^2\}, \tag{40}$$

and using ideas similar to that of Lemma 6, we have

$$\begin{aligned} |\tilde{I}_H^h u_H|^2_{H^1_h(\Omega_i)} &\preceq (1 + \log(H/h)) \{|u_H(P_1) - u_H(P_2)|^2 \\ &\quad + |u_H(P_2) - u_H(P_3)|^2 + |u_H(P_3) - u_H(P_1)|^2\} \\ &\asymp (1 + \log(H/h)) |u_H|^2_{H^1_h(\Omega_i)}. \end{aligned}$$

By construction, it is easy to see that

$$|(\tilde{I}_H^h u_H)(x)| \leq \max_{i=1,2,3} |u_H(P_i)|.$$

Therefore

$$\|\tilde{I}_H^h u_H - u_H\|^2_{L^2(\Omega_i)} \preceq \sum_i H^2 \, |u_H(P_i)|^2 \,,$$

and by using (36) and (25), we obtain (38).

Since, by assumption, $\rho(x) = \rho_i$ in each Ω_i, these arguments are also valid for the weighted norms and we obtain (37). □

Let us denote $\tilde{I}_h^{\tilde{\mathrm{F}}} = \tilde{I}_H^h I_h^H$. Using Lemmas 5 and 8 and the triangular inequality, we have:

Theorem 1. *Let $u \in \tilde{V}^h(\Omega)$. Then,*

$$(\tilde{I}_h^{\tilde{\mathrm{F}}} \tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^h(\Omega),$$

$$\|\tilde{I}_h^{\tilde{\mathrm{F}}} u - u\|_{L^2_\rho(\Omega_i)} \preceq H \, |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{41}$$

and

$$|\tilde{I}_h^{\tilde{\mathrm{F}}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + \log(H/h))^{\frac{1}{2}} \, |u|_{H^1_{\rho,h}(\Omega_i)}, \quad i = 1, \cdots, N. \tag{42}$$

Let $\theta^{\mathrm{CR}}_{\mathscr{E}_{ij}} \in \tilde{V}^h(\Omega_i)$, $j = 1, 2, 3$, be the nonconforming P_1 harmonic functions in Ω_i constructed with the values $\theta^{\mathrm{CR}}_{\mathscr{E}_{ij}} = 1$ at the CR nodes $\mathscr{E}^{\mathrm{CR}}_{ij,h}$, and $\theta^{\mathrm{CR}}_{\mathscr{E}_{ij}} = 0$ at the other CR nodes of $\partial\Omega^{\mathrm{CR}}_{i,h}$. It is easy to see that the $\theta^{\mathrm{CR}}_{\mathscr{E}_{ij}}$ form a basis of all nonconforming P_1 harmonic functions in Ω_i that take constant values on the edges of the substructure. Hence, the Interpolation Operator I_H^h given in Definition 3 can be given in terms of these $\theta^{\mathrm{CR}}_{\mathscr{E}_{ij}}$ functions. Let us denote the face interpolation operator by $\tilde{I}_h^{\mathrm{F}} = I_H^h I_h^H$. As in Theorem 1, we obtain

Theorem 2. *Let $u \in \tilde{V}^h(\Omega)$. Then*

$$\tilde{I}_h^{\mathrm{F}}(\tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^h(\Omega),$$

$$\|\tilde{I}_h^{\mathrm{F}} u - u\|_{L^2_\rho(\Omega_i)} \preceq H \, |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{43}$$

and

$$|\tilde{I}_h^{\mathrm{F}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + \log(H/h))^{\frac{1}{2}} \, |u|_{H^1_{\rho,h}(\Omega_i)} \quad i = 1, \cdots, N. \tag{44}$$

Proof. The inequality (44) follows trivially from (42) since the nonconforming P_1 harmonic function has minimal energy seminorm.

Using trivial arguments, we can show a weaker result than (43), given by

$$\|\tilde{I}_h^{\mathrm{F}} u - u\|_{L^2_\rho(\Omega_i)} \preceq H \, (1 + \log(H/h))^{\frac{1}{2}} |u|_{H^1_{\rho,h}(\Omega_i)}. \tag{45}$$

We note that (45) will be enough for our purposes. To prove (43), we use the same ideas as in the proof of Lemma 4.3 of Dryja, Smith, and Widlund [14]; we note that the convexity of Ω_i is used. □

Remark 2. It is easy to see that we do not need to use the fact that $u_H \in V_H(\Omega)$; we only need to calculate the values $\bar{u}_{\mathscr{E}_{ij}} = u_H(P_{ij})$ by Definition 2. The next step is to provide the constant value $\bar{u}_{\mathscr{E}_{ij}}$ to all CR nodes on $\mathscr{E}_{ij,h}^{\mathrm{CR}}$ and perform a discrete nonconforming P_1 harmonic extension, or an approximate harmonic extension. An important observation is that these extensions can be constructed for nontriangular substructures. In a case of approximate harmonic extension, in a first step, we construct a partition of unity in Ω_i. This can be done by using ideas similar to those of the triangular case, if Ω_i is not too degenerate. By using the same technique as in the proof of Lemma 8, we can show that

$$|\tilde{I}_h^{\tilde{\mathrm{F}}} u|^2_{H^1_{\rho,h}(\Omega_i)} \preceq \rho_i \,(1+\log(H/h)) \sum_{\mathscr{E}_{ij}\subset\partial\Omega_i} (\bar{u}_{\mathscr{E}_{ij}} - \bar{u}_{\partial\Omega_i})^2. \tag{46}$$

Here, $\bar{u}_{\partial\Omega_i}$ is the average of u over $\partial\Omega_i$. We obtain (42) by showing that each term in the sum can be bounded by $C\,|u|^2_{H^1_{\rho,h}(\Omega_i)}$.

7. The Neumann-Neumann basis

We consider Neumann-Neumann coarse spaces. This is the nonconforming P_1 version of a coarse space studied in Dryja and Widlund [18], and Mandel and Brezina [20]. However, here we use an approximate harmonic extension inside the substructures. We note that the coarse spaces considered by these authors differ only in how certain weights are chosen; Mandel and Brezina use weights that are convex combinations of the coefficient $\rho(x)$, while Dryja and Widlund use $\rho^{\frac{1}{2}}(x)$. Here we show that any convex combination of the $\rho^{\beta}(x)$, for $\beta \geq 1/2$, leads to stability. We remark that we can even define a Neumann-Neumann coarse space for $\beta = \infty$ by considering the limit of $\beta \to \infty$; see Dryja, Sarkis, and Widlund [13]. We point out that the choice $\beta = 1/2$ can be viewed as a L^2-average, while $\beta = 1$ is an average in the L^1 sense.

The coarse spaces of the previous section are *face based*. There are some differences between Neumann-Neumann and face based coarse spaces. A Neumann-Neumann coarse spaces has only one degree of freedom per substructure, while a face based uses one degree of freedom per edge. A Neumann-Neumann basis function associated with the substructure Ω_i, has support in Ω_i and its neighboring substructures, while a face based function basis, associated with an edge of a substructure, has support in just two substructures. The face based coarse spaces appear to be more stable since all the estimates, related to the jumps of the coefficients, are tight. In the lemmas that we have proved for the face based methods, all the stability results were derived in individual substructures, while in the Neumann-Neumann cases, we need to work with the neighboring substructures as well.

Definition 4. The Neumann-Neumann interpolation operator, $\widetilde{\tilde{I}_h^{NN}} : \tilde{V}_0^h \to \tilde{V}_0^h$, is given as follows:

i) For each substructure Ω_i, calculate the mean value on $\partial\Omega_i$, i.e.

$$\bar{u}_i = \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} u(s)\, ds.$$

Here $|\partial\Omega_i|$ is the length of $\partial\Omega_i$.

ii) Let P_{ij} be the midpoint of the edge $\mathscr{E}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$. Then for all CR node $P \in \mathscr{E}_{ij,h}^{\mathrm{CR}}$, let $\widetilde{\tilde{I}_h^{NN}} u(P) = (\hat{I}_h^H u)(P_{ij})$, where

$$(\hat{I}_h^H u)(P_{ij}) = \frac{\rho_i^\beta}{\rho_i^\beta + \rho_j^\beta}\, \bar{u}_i + \frac{\rho_j^\beta}{\rho_i^\beta + \rho_j^\beta}\, \bar{u}_j.$$

iii) Let P_{ij} be the midpoint of the edge $\mathscr{E}_{ij}$ common to $\bar{\Omega}_i$ and $\partial\Omega$. Then for all CR node $P \in \mathscr{E}_{ij,h}^{\mathrm{CR}}$, let $\widetilde{\tilde{I}_h^{NN}} u(P) = (\hat{I}_h^H u)(P_{ij})$, where

$$(\hat{I}_h^H u)(P_{ij}) = (I_h^H u)(P_{ij}).$$

iv) Perform an approximate harmonic extension to define $\widetilde{\tilde{I}_h^{NN}} u$ in the interior of the substructures.

Note that we can also calculate $\bar{u}_i$ by:

$$\bar{u}_i = \sum_{j=1}^{3} \frac{|\mathscr{E}_{ij}|}{|\partial\Omega_i|} (I_h^H u)(P_{ij}). \tag{47}$$

Therefore, there exists a linear transformation $I_H^H : \tilde{V}^H(\Omega) \to \tilde{V}^H(\Omega)$, such that $\hat{I}_h^H u = I_H^H I_h^H u$. The next lemma establishes stability properties for I_H^H.

Lemma 9. *Let $u_H \in \tilde{V}^H(\Omega)$ and $1/2 \le \beta \le \infty$. Then*

$$|I_H^H u_H|_{H^1_{\rho,H}(\Omega_i)} \preceq |u_H|_{H^1_{\rho,H}(\Omega_i^{\mathrm{ext}})}, \tag{48}$$

and

$$\|I_H^H u_H - u_H\|_{L^2_\rho(\Omega_i)} \preceq H\, |u_H|_{H^1_{\rho,H}(\Omega_i^{\mathrm{ext}})}. \tag{49}$$

Here the extended domain Ω_i^{ext} is the union of Ω_i and the substructures that share an edge with Ω_i.

Proof. Let us first prove the L^2_ρ error stability. We focus on the case where P_{ij} does not belongs to $\partial\Omega$. The other case is trivial. Note that (see Fig. 4)

$$|u_H(P_{ij}) - (I_H^H u_H)(P_{ij})|^2 = |u_H(P_{ij}) - \frac{{\rho_i}^\beta\, \bar{u}_i + \rho_j^\beta\, \bar{u}_j}{\rho_i^\beta + \rho_i^\beta}|^2.$$

By using (47) and simple calculations, this quantity is equal to

$$\frac{1}{|\rho_i^\beta + \rho_j^\beta|^2} *$$

$$|\rho_i^{\beta}\{\frac{|\mathscr{E}_{ik}|}{|\partial\Omega_i|}(u_H(P_{ij})-u_H(P_{ik}))+\frac{|\mathscr{E}_{il}|}{|\partial\Omega_i|}(u_H(P_{ij})-u_H(P_{il}))\}$$

$$+\rho_j^{\beta}\{\frac{|\mathscr{E}_{js}|}{|\partial\Omega_j|}(u_H(P_{ij})-u_H(P_{js}))+\frac{|\mathscr{E}_{jr}|}{|\partial\Omega_j|}(u_H(P_{ij})-u_H(P_{jr}))\}|^2.$$

Using the shape regularity of the subdomains, it is easy to see that

$$\rho_i\,|u_H(P_{ij})-(I_H^H u_H)(P_{ij})|^2 \tag{50}$$

$$\preceq \frac{\rho_i^{2\beta}}{|\rho_i^{\beta}+\rho_j^{\beta}|^2}\,|u_H|^2_{H^1_{\rho,H}(\Omega_i)}+\frac{\rho_i\,\rho_j^{2\beta-1}}{|\rho_i^{\beta}+\rho_j^{\beta}|^2}\,|u_H|^2_{H^1_{\rho,H}(\Omega_j)},$$

and using the fact that $\beta \geq 1/2$, we can bound this quantity by

$$\preceq |u_H|^2_{H^1_{\rho,H}(\Omega_i\cup\Omega_j)}.$$

We note that the constant related to the last bound, is also independent of β. Therefore, our results also holds for $\beta = \infty$. We obtain (49) by adding all the contributions (50) to the $L^2_\rho(\Omega_i)$ norm. We prove (48) by using the triangular inequality, an inverse inequality, and (49). □

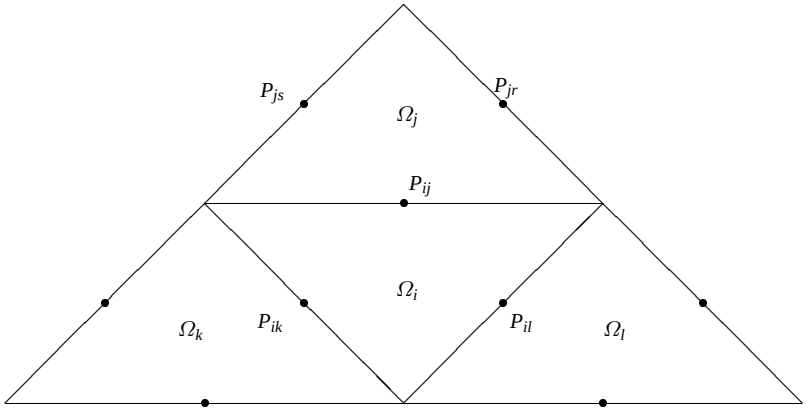

Fig. 4.

Theorem 3. *Let $u \in \tilde{V}^h(\Omega)$ and $1/2 \leq \beta \leq \infty$. Then*

$$\|\widetilde{\tilde{I}_h^{NN}}u-u\|_{L^2_\rho(\Omega_i)} \preceq H\,|u|_{H^1_{\rho,h}(\Omega_i^{\mathrm{ext}})}, \tag{51}$$

and

$$|\widetilde{\tilde{I}_h^{NN}}u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1+\log(H/h))^{\frac{1}{2}}\,|u|_{H^1_{\rho,h}(\Omega_i^{\mathrm{ext}})}. \tag{52}$$

Proof. Using Lemmas 8, 9, and 7, we have

$$|\widetilde{\tilde{I}_h^{NN}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1+\log(H/h))^{\frac{1}{2}} \, |I_H^H I_h^H u|_{H^1_{\rho,H}(\Omega_i)}$$

$$\preceq (1+\log(H/h))^{\frac{1}{2}} \, |I_h^H u|_{H^1_{a,H}(\Omega_i^{\text{ext}})}$$

$$\preceq (1+\log(H/h))^{\frac{1}{2}} \, |u|_{H^1_{\rho,h}(\Omega_i^{\text{ext}})}.$$

The L^2_ρ error stability is obtained by

$$\|\widetilde{\tilde{I}_h^{NN}} u - u\|_{L^2_\rho(\Omega_i)} \leq \|\widetilde{\tilde{I}_h^{NN}} u - I_H^H I_h^H u\|_{L^2_\rho(\Omega_i)} +$$

$$\|I_H^H I_h^H u - I_h^H u\|_{L^2_\rho(\Omega_i)} + \|I_h^H u - u\|_{L^2_\rho(\Omega_i)},$$

and by using Lemmas 8, 9, and 7. □

Remark 3. Using the same ideas in Theorem 2 the analysis and results in this section can also be extended to the case in which we replace in step iv) of Definition 4 the approximate harmonic extension by the discrete nonconforming P_1 harmonic extension inside each Ω_i. We denote the corresponding interpolator by $\tilde{I}_h^{NN}$.

8. The three-dimensional case

We show in this section that the methods developed above can be extended to three dimensions.

For simplicity, we assume that Ω is a polyhedral region of diameter 1 in three dimensional space. As before, we introduce a nonoverlapping partition composed of tetrahedra Ω_i of diameter of order H. This defines a coarse space and a triangulation $\mathscr{T}^H$. We further subdivide the substructures into tetrahedra which results in a triangulation $\mathscr{T}^h$ and define the nonconforming P_1 finite element spaces $\tilde{V}^h(\Omega)$, $\tilde{V}_0^h(\Omega)$, $\tilde{V}^H(\Omega)$, $\tilde{V}_0^H(\Omega)$ as the three-dimensional counterpart of Definition 1. Here, the continuity is enforced at the barycenter of the faces of the triangulations.

The local equivalence map is given by the following procedure. In each tetrahedral element of $\mathscr{T}^h$ (cf. Fig. 5), we connect the centroid to the four vertices and to the barycenters of the four faces. We also connect each face barycenter to the three vertices of the face. Thus, we subdivide each tetrahedral element into twelve subtetrahedra. We denote this new triangulation by $\mathscr{T}^{\hat{h}}$. The vertices of $\mathscr{T}^{\hat{h}}$ are the vertices, barycenters, and centroids of the elements of $\mathscr{T}^h$. Denote by $V^{\hat{h}}(\Sigma)$ the conforming space of continuous, piecewise linear functions on the triangulation $\mathscr{T}^{\hat{h}}$ restricted to a region Σ. We denote by CR nodal points, the barycenters of the faces of elements in $\mathscr{T}^h$. We use notations similar to those used for two dimensions. $\mathscr{F}_{ij,h}^{\text{CR}}$, $\partial\Omega_{i,h}^{\text{CR}}$, and $\overline{\Omega}_{i,h}^{\text{CR}}$ represent the sets of CR nodal points belonging to a face $\mathscr{F}_{ij}$, $\partial\Omega_i$, and $\overline{\Omega}_i$, respectively.

$\bar{\Omega}_{i,\hat{h}}$ and $\partial\Omega_{i,\hat{h}}$ denote the sets of vertices of $\mathscr{T}^{\hat{h}}$ belonging to $\bar{\Omega}_i$, and $\partial\Omega_i$, respectively. $\partial\Omega_{i,h}$ represents the set of vertices of $\mathscr{T}^h$ belonging to $\partial\Omega_i$.

The counterpart of Isomorphism 1 is given by the following local equivalence map $\mathscr{M}_i^{\mathscr{F}_{ij}} : \tilde{V}^h(\Omega_i) \to V^{\hat{h}}(\Omega_i)$, by:

Isomorphism 2. Given $u \in \tilde{V}^h(\Omega_i)$ and a face $\mathscr{F}_{ij}$ of $\partial\Omega_i$, define $\mathscr{M}_i^{\mathscr{F}_{ij}} u \in V^{\hat{h}}(\Omega_i)$ by the values $\mathscr{M}_i^{\mathscr{F}_{ij}} u$ at the following sets of points:

i) If $P \in \bar{\Omega}_{i,\hat{h}} \backslash \partial\Omega_{i,\hat{h}}$ and the $\tau_j^{\hat{h}}$ are the elements in $\mathscr{T}^{\hat{h}}(\Omega_i)$ that have P as a vertex, then

$$\mathscr{M}_i^{\mathscr{F}_{ij}} u(P) = \text{mean of} \quad u|_{\tau_j^{\hat{h}}}(P).$$

Here $u|_{\tau_j^{\hat{h}}}(P)$ is the limit of $u(x)$ when $x \in \tau_j^{\hat{h}}$ approaches P.

ii) If $P \in \partial\Omega_{i,h}^{\mathrm{CR}}$, then

$$\mathscr{M}_i^{\mathscr{F}_{ij}} u(P) = u(P).$$

iii) Let $P \in \partial\Omega_{i,h} \cap \partial\mathscr{F}_{ij}$, and let T_j, $j = 1, \cdots, N_P^{\mathscr{F}_{ij}}$, be the triangles of $\mathscr{T}^h \cap \mathscr{F}_{ij}$ that have P as a vertex. Then

$$\mathscr{M}_i^{\mathscr{F}_{ij}} u(P) = \sum_{k=1}^{N_P^{\mathscr{F}_{ij}}} \frac{|T_k|}{\cup_{j=1}^{N_P^{\mathrm{F}}} |T_j|}\, u(Q_i).$$

Here Q_i and $|T_i|$ are the barycenter and the area of the triangle T_i, respectively.

iv) Let $P \in \partial\Omega_{i,h} \backslash \partial\mathscr{F}_{ij}$, and let T_j, $j = 1, \cdots, N_P$, be the triangles of $\mathscr{T}^h \cap \partial\Omega_i$ that have P as a vertex. Then

$$\mathscr{M}_i^{\mathscr{F}_{ij}} u(P) = \sum_{k=1}^{N_P} \frac{|T_k|}{\cup_{j=1}^{N_P} |T_j|}\, u(Q_i).$$

Remark 4. It is easy to check that Lemma 3 holds, if we replace $V^{h/2}(\Omega_i)$ by $V^{\hat{h}}(\Omega_i)$, and let the faces play the role previously played by the edges.

Definition 5. Let $v \in \tilde{V}^h(\Omega)$. The Interpolation operator $I_h^H : \tilde{V}^h(\Omega) \to \tilde{V}^H(\Omega)$, is given by:

i) If P_{ij} is the barycenter of the face $\mathscr{F}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, then

$$(I_h^H v)(P_{ij}) = \frac{1}{|\mathscr{F}_{ij}|} \int_{\mathscr{F}_{ij}} v|_{\bar{\Omega}_i}(x)\, dx = \frac{1}{|\mathscr{F}_{ij}|} \int_{\mathscr{F}_{ij}} v|_{\bar{\Omega}_j}(x)\, dx.$$

ii) If P_{ij} is the barycenter of a face $\mathscr{F}_{ij}$ common to $\bar{\Omega}_i$ and $\partial\Omega$, then

$$(I_h^H v)(P_{ij}) = \frac{1}{|\mathscr{F}_{ij}|} \int_{\mathscr{F}_{ij}} v|_{\bar{\Omega}_i}(x)\, dx.$$

Using the same ideas as in two dimensions, we can prove lemmas analogous to Lemmas 5-7.

Definition 6. Let the prolongation operators $I_H^h, \tilde{I}_H^h : \tilde{V}^H(\Omega) \to \tilde{V}^h(\Omega)$, be defined as in the two dimensional case. In a first step, let define $(I_H^h u_H)(q) = (\tilde{I}_H^h u_H)(q) = u_H(C_{ij})$ for all CR nodes $q \in \mathscr{F}_{ij,h}^{\mathrm{CR}}$. Finally, perform a nonconforming P_1 harmonic or approximate harmonic extension.

We describe the three dimensional version of Extension 1. Let P_j, $j = 1, \cdots, 4$, be the barycenters of the faces of $\partial\Omega_i$, and let V_j be the vertex of Ω_i that is opposite to P_j. Let C the centroid of Ω_i, i.e. the intersection of the line segments connecting the V_j to the P_j. Let E_{jk}, $k = 1, 2, 3$, be the edges of $\partial\mathscr{F}_j$; see Fig. 5.

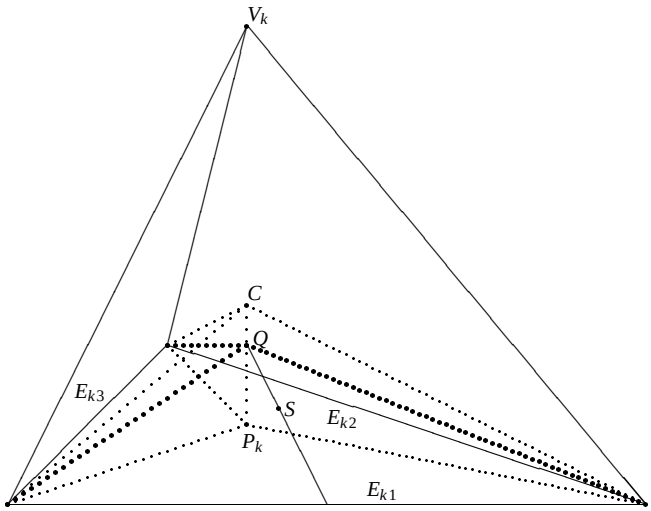

Fig. 5.

Extension 2. The construction of an approximate harmonic extension in Ω_i is defined by the following steps (see Fig. 5):

i) Let

$$\bar{u}(C) = \frac{1}{4} \sum_{j=1}^{4} u_H(P_j).$$

ii) For a point Q that belongs to a line segment connecting C to P_j, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(P_j)$, i.e. by

$$\bar{u}(Q) = \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(P_j).$$

Here $\lambda(Q) = \mathrm{dist}(Q, P_j)/\mathrm{dist}(C, P_j)$.

iii) For a point S that belongs to any of the three triangles defined by the previous Q, and the edges $\mathscr{E}_{jk}$, $k = 1, \cdots, 3$, let

$$\bar{u}(S) = \bar{u}(Q).$$

iv) Finally, let $I_H^h u_H(\Omega_i) = I_{i,h}^{\mathrm{CR}} \bar{u}$, where $I_{i,h}^{\mathrm{CR}}$ is the interpolation operator into the space $\tilde{V}^h(\Omega_i)$ that preserves the values of a function at the CR nodes of $\bar{\Omega}_{i,h}^{\mathrm{CR}}$.

The prolongation operator $\tilde{I}_H^h$ in three dimensions has the same stability properties as in the two dimensional case, i.e. Lemma 8 still holds. The idea of the proof is the following. Consider the case where $u_H(\Omega_i)$ is given by $u_H(P_1) = 1$ and $u_H(P_2) = u_H(P_3) = u_H(P_4) = 0$. This gives an element of the partition of unity introduced by Dryja, Smith, and Widlund [14] and the energy seminorm of u_H is on the order of H. Let $\vartheta_{\mathscr{F}_{i1}}^{\mathrm{CR}}(\Omega_i) = I_H^h u_H(\Omega_i)$. We note that $|\nabla \vartheta_{\mathscr{F}_{i1}}^{\mathrm{CR}}(x)|$ is bounded by C/r, where r is the distance to the nearest edge of Ω_i. The contribution to the energy seminorm from the union of the elements with at least one vertex on the edge of the substructure can be bounded by CH, using that the extension is given by a convex combination of the boundary values. To estimate the contribution to the energy from the rest of the substructure, we introduce cylindrical coordinates using the appropriate substructure edge as the z-axis. Integrating $|\nabla \vartheta_{\mathscr{F}_{i1}}^{\mathrm{CR}}(x)|^2$ over this region, we find that it is bounded by $C\,(1 + \log(H/h))\,H$. See Dryja, Smith, and Widlund [14] for more details; the arguments can be extended easily for the nonconforming P_1 case.

To prove Lemma 8 for a general $u_H \in \tilde{V}^H(\Omega_i)$, we use the same ideas as for two dimensions.

As in the two-dimensional case, we can extend the results to nontriangular substructures and to the Neumann-Neumann case.

9. Main result for two-level method

In this section, we consider the Schwarz method introduced in the previous subsections and prove the following result.

Theorem 4. *The operator $\tilde{P}$ of the additive Schwarz algorithm, defined by the spaces $\tilde{V}_0^h(\Omega)$ and $\tilde{V}_i$, satisfies:*

$$\kappa(\tilde{P}) \preceq (1 + \log(\frac{H}{h}))\,(1 + \frac{H}{\delta}). \tag{53}$$

Here $\kappa(\tilde{P})$ is the condition number of $\tilde{P}$, and Ω is a region in $\Re^2$ or in $\Re^3$. Therefore, if we use a generous overlapping, i.e. H/δ is uniformly bounded, then

$$\kappa(\tilde{P}) \preceq 1 + \log(\frac{H}{h}).$$

Proof. The lower bound is obtained by using *Assumption* i) of Lemma 2. We partition the finite element function $u \in \tilde{V}_0^h(\Omega)$, as follows. We first choose $u_0 = \tilde{I}_h^X u$, with $X = F$, $\tilde{F}$, NN, or $\widetilde{NN}$, and set $w = u - u_0$. The other terms in the representation of u are defined by $u_i = I_h^{\mathrm{CR}}(\theta_i w), i = 1, \cdots, N$. Here I_h^{CR} is the linear interpolation operator into the space $\tilde{V}_0^h(\Omega)$ that preserves the values at the CR nodes of Ω_h^{CR}, and $\{\theta_i\}$ is a partition of unity with $\theta_i \in C_0^\infty(\Omega_i')$ and $\sum \theta_i(x) = 1$.

For a relatively generous overlap of the subdomains, these functions can be chosen so that $\nabla\theta_i$ is bounded by C/H. By using the linearity of I_h^{CR}, we can show that we have a correct partition of u. In order to estimate the seminorm of u_i, we work on one element τ_j^h at a time. We obtain

$$|u_i|^2_{H^1_{\rho,h}(\tau_j^h)} \leq 2\,|\overline{\theta_i} w|^2_{H^1_{\rho,h}(\tau_j^h)} + 2\,|I_h^{\mathrm{CR}}((\theta_i - \overline{\theta_i})w)|^2_{H^1_{\rho,h}(\tau_j^h)}.$$

Here $\overline{\theta_i}$ is the average value of θ_i over τ_j^h. Here and afterwards we use that the coefficient $\rho(x)$ is constant in each element τ_j^h. It is easy to see, by using the inverse inequality (9), that

$$|I_h^{\mathrm{CR}}((\theta_i - \overline{\theta_i})w)|^2_{H^1_{\rho,h}(\tau_j^h)} \preceq h^{-2}\,\|I_h^{\mathrm{CR}}((\theta_i - \overline{\theta_i})w)\|^2_{L^2_\rho(\tau_j^h)}.$$

We can now use the fact that on τ_j^h, θ_i differs from its average by at most $C\,h/H$. After summing over all elements of Ω_i', we arrive at the inequality

$$|u_i|^2_{H^1_{\rho,h}(\Omega_i')} \preceq |w|^2_{H^1_{\rho,h}(\Omega_i')} + H^{-2}\,\|w\|^2_{L^2_\rho(\Omega_i')}. \tag{54}$$

We sum over all i and use that each point in Ω is covered only a fixed number of times and we obtain a uniform bound on C_0^2. We conclude the proof, by estimating the two terms of

$$|w|^2_{H^1_{\rho,h}(\Omega)} + H^{-2}\,\|w\|^2_{L^2_\rho(\Omega)}$$

by $|u|^2_{H^1_{\rho,h}(\Omega)}$. The bounds follow by using the stability results of Theorems 1, 2, 3, or the Remark 3, or the results of Sect. 8.

For the case of small overlap, the proof is similar to that of the case of piecewise linear conforming space considered in Dryja and Widlund [19]. Let $\Gamma = \bigcup \partial\Omega_i \backslash \partial\Omega$. Let $\Gamma_{\delta,i} \subset \Omega_i$ be the region that is within a distance δ of Γ. In order to use Lemma 2, we first estimate $|u_i|_{H^1_{\rho,h}(\Omega)}$ in terms of $|w|_{H^1_{\rho,h}(\Omega)}$. We consider the contribution from one substructure at a time and we obtain

$$|u_i|^2_{H^1_{\rho,h}(\Omega_i \backslash \Gamma_{\delta,i})} = |w|^2_{H^1_{\rho,h}(\Omega_i \backslash \Gamma_{\delta,i})},$$

and

$$|u_i|^2_{H^1_{\rho,h}(\Gamma_{\delta,i})} \preceq |w|^2_{H^1_{\rho,h}(\Gamma_{\delta,i})} + \delta^{-2}\,\|w\|^2_{L^2_\rho(\Gamma_{\delta,i})}.$$

We also need to estimate $|u_j|^2_{H^1_{\rho,h}(\Gamma_{\delta,i})}|$ for the j that correspond to neighboring substructures. This presents no new difficulties. It remains to estimate $\|w\|^2_{L^2_\rho(\Gamma_{\delta,i})}$. It is easy to check that

$$\|w\|_{L^2_\rho(\Gamma_{\delta,i})} \preceq \|\mathscr{M}_i^{\mathscr{E}_{ij}} w\|_{L^2_\rho(\Gamma_{\delta',i})},$$

where $\Gamma_{\delta',i} \subset \Omega_i$ is the region obtained by extending $\Gamma_{\delta,i}$ within a distance of two elements of $\partial\Gamma_{\delta,i}$. Using Lemma 3 of Dryja and Widlund [19] and Lemma 3, we obtain

$$\|\mathscr{M}_i^{\mathscr{E}_{ij}} w\|^2_{L^2_\rho(\Gamma_{\delta',i})} \preceq \delta'^2((1 + H/\delta')|\mathscr{M}_i^{\mathscr{E}_{ij}} w|^2_{H^1_\rho(\Omega_i)} + 1/H\delta'\|\mathscr{M}_i^{\mathscr{E}_{ij}} w\|_{L^2_\rho(\Omega_i)})$$

$$\preceq (\delta)^2((1 + H/\delta)|w|^2_{H^1_{\rho,h}(\Omega_i)} + 1/H\delta\ \|w\|_{L^2_{\rho,h}(\Omega_i)}.$$

We note that we have used that the coefficient $\rho(x)$ is nearly constant in Ω_i. We note that each $x \in \Omega$ is covered only a finite number of times by the subregions. The rest of the proof is similar to that of generous overlap case given above.

An upper bound on the spectrum is obtained by bounding

$$a^h(\tilde{P}v, v) = a^h(\tilde{P}_0 v, v) + a^h(\tilde{P}_1 v, v) + \cdots + a^h(\tilde{P}_N v, v), \tag{55}$$

from above in terms of $a^h(v, v)$. Using Schwarz's inequality, the fact that the $\tilde{P}_i$ are projections, and that the maximum number of regions that intersect at any point is uniformly bounded, it is easy to show that the spectrum of $\tilde{P}$ is bounded from above by

$$\max_{p \in\ \Omega} \{\#(i : p \in\ \Omega_i') + 1\}. \quad \square$$

Remark 5. The proof of Theorem 4 also works for triangulations $\mathscr{T}^H$ and $\mathscr{T}^h$ that are shape regular and nonuniform. We can then show that

$$\kappa(\tilde{P}) \preceq \max_{\{1,\cdots,N\}} (1 + \log(\frac{H_i}{h_i}))\,(1 + \frac{H_i}{\delta_i}).$$

Here, H_i is the diameter of Ω_i, h_i is the diameter of the smallest element in $\mathscr{T}^h(\Omega_i)$, and δ_i measures the overlap.

Acknowledgements I would like to thank my thesis advisor, Olof Widlund, for all his guidance, and time he has been devoting to my work. This paper is an extended and refined version of part of my doctoral dissertation. The author is also indebted to the professor Max Dryja for many suggestions.

References

1. Arnold, D.N., Brezzi, F. (1985): Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. Math. Model. and Numer. Anal., **19**, 7–32
2. Brenner, S.C. (1994): A two-level additive Schwarz Preconditioner for nonconforming plate elements. Preprint
3. Brenner, S.C. (1994): Two-level additive Schwarz preconditioners for nonconforming finite element methods, Seventh International Conference of Domain Decomposition Methods in Scientific and Engineering Computing. Keyes, D.E., Xu, J. (eds.), AMS, Held at Penn State University, 1993
4. Brenner, S.C. (1989) An optimal-order multigrid method for P1 nonconforming finite elements. Math. Comp., **53** (89), 1–15
5. Cai, X.-C., The use of pointwise interpolation in domain decomposition methods with non-nested meshes, SIAM J. Sci. Comput., to appear

6. Cai, X.-C. (1993): A non-nested coarse space for Schwarz type domain decomposition methods. Department of Computer Science, University of Colorado at Boulder
7. Chan, T.F., Smith, B., Zou, J. (1994): Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. Tech. Report CAM 94-08, Department of Mathematics, UCLA
8. Ciarlet, P.G. (1978): The Finite Element Method for Elliptic Problems. North-Holland
9. Cowsar, L.C. (1993): Domain decomposition methods for nonconforming finite elements spaces of Lagrange-type, Volume 1. In: Melson, N.D., Manteuffel, T.A., McCormick, S.F. (eds.) Proceedings of the Sixth Copper Mountain Conference on Multigrid Methods, **3224** 93–109, Hampton VA NASA
10. Cowsar, L.C. (1994): Some Domain Decomposition and Multigrid Preconditioners for Hybrid Mixed Finite Elements. PhD thesis, Rice University
11. Cowsar, L.C., Mandel, J., Wheeler, M.F. (1993): Balancing Domain Decomposition for Mixed Finite Elements. Tech. Report TR93-08, Department of Mathematical Sciences, Rice University, to appear in Math. Comp.
12. Crouzeix, M., Raviart, P.A. (1973): Conforming and non-conforming finite element methods for solving the stationary Stokes equations. RAIRO Anal. Numer., **7** 33–76
13. Dryja M., Sarkis, M., Widlund, O.B. (1996): Multilevel Schwarz methods for elliptic problems with discontinuous coefficients in three dimensions. Department of Computer Science, Courant Institute, Numer. Math., **72** 313–348
14. Dryja, M., Smith, B.F., Widlund, O.B. (1994): Schwarz Analysis of Iterative Substructuring Algorithms for Elliptic Problems in Three Dimensions. SIAM J. Numer. Anal. **31**, 1662–1694
15. Dryja, M., Widlund, O.B. (1987): An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions. Tech. Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute
16. Dryja, M., Widlund, O.B. (1989): Some Domain Decomposition Algorithms for Elliptic Problems. In: Hayes, L., Kincaid D. (eds.) Iterative Methods for Large Linear Systems. San Diego, California, Academic Press, pp. 273–291. Proceeding of the Conference on Iterative Methods for Large Linear Systems held in Austin, Texas, October 19–21, 1988, to celebrate the sixty-fifth birthday of David M. Young, Jr.
17. Dryja, M., Widlund, O.B. (1990): Towards a Unified Theory of Domain Decomposition Algorithms for Elliptic Problems. In: Chan, T., Glowinski, R., Périaux, J., Widlund, O. (eds.) Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20-22, 1989, SIAM, Philadelphia, PA
18. Dryja, M., Widlund, O.B. (1993): Schwarz Methods of Neumann-Neumann Type for Three-Dimensional Elliptic Finite Element Problems. Tech. Report 629, Department of Computer Science, Courant Institute. To appear in Comm. Pure Appl. Math.
19. Dryja, M., Widlund, O.B. (1994): Domain Decomposition Algorithms with Small Overlap, SIAM J. Sci. Comput., **15**, 604–620
20. Mandel, J., Brezina, M. (1992): Balancing Domain Decomposition: Theory and Computations in Two and Three Dimensions. Computational Mathematics Group, University of Colorado at Denver
21. Matsokin A.M., Nepomnyaschikh, S.V. (1985): A Schwarz alternating method in a subspace. Soviet Mathematics, **29**(10), 78–84
22. Nečas, J. (1967): Les Méthodes Directes en Théorie des Equations Elliptiques. Academia, Prague
23. Oswald, P. (1992): On a hierarchical basis multilevel method with nonconforming P1 elements. Numer. Math., To appear
24. Sarkis, M. (1993): Two-Level Schwarz methods nonconforming finite elements and discontinuous coefficients. Tech. Report 629, Department of Computer Science, Courant Institute
25. Sarkis, M. (1993): Multilevel Methods for P_1 Nonconforming Finite Elements and Discontinuous Coefficients in Three Dimensions. In: Keyes, D.E., Xu, J. (eds.) Seventh International Conference of Domain Decomposition Methods in Scientific and Engineering Computing, AMS, Held at Penn State University, October 27–30
26. Sarkis, M. (1994): Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Conforming and Non-Conforming Elements. Tech. Report 671, PhD thesis, Courant Institute of Mathematical Sciences, Department of Computer Science, Courant Institute
27. Zhang, X. (1992): Multilevel Schwarz Methods, Numer. Math. **63**, 521–539