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Multilevel Methods for P_1 Nonconforming Finite Elements and Discontinuous Coefficients in Three Dimensions

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ABSTRACT. We introduce multilevel Schwarz preconditioners for solving the discrete algebraic equations that arise from a nonconforming P_1 finite element method approximation of second order elliptic problems. For the additive multilevel version, we obtain a condition number bounded by $C_1\,(1+\log H/h)^2$ from above, and for the multiplicative versions, such as the V-cycle multigrid methods using Gauss Seidel and damped Jacobi smoothers, we obtain a rate of convergence bounded from above by $1 - C_2\,(1+\log H/h)^{-2}$.

1. Introduction

There are many engineering applications in which the main goal is to find a good approximation for $q = \rho\,\nabla u$. Here, u is the solution of an elliptic problem with coefficient ρ. We can find an approximation for q by finding an approximation for u and then applying the operator $\rho\,\nabla$. This procedure may generate serious errors since when ρ becomes more discontinuous, the solution u becomes more singular and the operator $\rho\,\nabla$ more numerically unstable. Furthermore, we note that in the interior of Ω we have, formally, $\operatorname{div} q = f$. Therefore, we expect $q(x)$ to be less sensitive than $u(x)$ to variations of $\rho(x)$. For instance, if we consider the one-dimensional case with $f = 0$ and inhomogeneous Dirichlet data, we obtain $q =$ constant. This is why mixed methods are introduced in order to approximate $\rho\,\nabla u$ and u, independently. Our motivation for considering the P_1-nonconforming space comes from the fact that there is an equivalence

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between mixed methods and nonconforming methods; see Arnold and Brezzi [**1**].

The first multigrid methods for nonconforming finite elements were introduced by Braess and Verfüth [**2**], and Brenner [**3**]. The existing convergence results are based on the H^2–regularity assumption for the continuous problem. Later, Oswald [**12**], Vassilevski and Wang [**15**] proposed optimal multilevel BPX-preconditioners for nonconforming P_1 elements in three-dimensional case by using a sequence of nested conforming subspaces. No additional regularity assumption beyond the H^1 is used. We note, however, that we cannot guarantee that the rate of convergence of these methods are insensitive to large variations in the coefficients of the differential equation; see also [**11**], [**16**], and [**10**]. In this paper, we modify the Oswald preconditioner by introducing our nonstandard coarse spaces Sarkis [**13**], and establish that its condition number grows at most as the square of the number of levels, and does not depend on the number of substructures and the jumps of the coefficients. To analyze our methods, we introduce nonstandard local interpolators [**13, 14**] in order to convert results from the conforming to the nonconforming case. We note that an operator similar to ours has independently been introduced in Cowsar, Mandel, and Wheeler [**4**].

This paper is very closely related to those of Dryja [**6**], and Dryja, Sarkis, and Widlund [**8**] and we refer to [**6**] for some of our notation. The proofs of our results can be found in [**14**].

2. Notation

A coarse triangulation of Ω is introduced by dividing the region into nonoverlapping simplicial substructures Ω_i, $i = 1, \cdots, N$, with diameters of order H. The barycenters of faces of tetrahedra $\tau_j^h \in \mathcal{T}^h$ are called CR nodal points. The sets of CR nodal points belonging to $\bar{\Omega}$, $\partial\Omega$, $\mathcal{F}_{ij}$, $\partial\Omega_i$, and Γ are denoted by Ω_h^{CR}, $\partial\Omega_h^{CR}$, $\mathcal{F}_{ij,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, and Γ_h^{CR}, respectively.

Definition 1. *The nonconforming P_1 element spaces on the h-mesh (cf. Crouzeix and Raviart [**5**]) are given by*

$$\tilde{V}^h(\Omega) := \{v \,|\, v \text{ linear in each tetrahedron } \tau_j^h \in \mathcal{T}^h, \text{ and } v \text{ continuous at the nodes of } \Omega_h^{CR}\}, \text{ and}$$

$$\tilde{V}_0^h(\Omega) := \{v \,|\, v \in \tilde{V}^h(\Omega) \text{ and } v = 0 \text{ at the nodes of } \partial\Omega_h^{CR}\}.$$

Note that $\tilde{V}_0^h(\Omega)$ is nonconforming since $\tilde{V}_0^h(\Omega) \not\subset H_0^1(\Omega)$.

For $u \in \tilde{V}_0^h(\Omega)$, we define a nonconforming discrete weighted energy norm with $\rho = \rho_i$ on Ω_i by

$$|u|^2_{H^1_{\rho,h}(\Omega)} := a^h(u,u)\,, \tag{1}$$

where

$$a^h(u,v) = \sum_{i=1}^{N} \sum_{\tau_j^h \in \Omega_i} \int_{\tau_j^h} \rho_i \, \nabla u \cdot \nabla v \, dx = \sum_{i=1}^{N} a_{\Omega_i}^h(u,w). \tag{2}$$

Our discrete problem is given by:
Find $u \in \tilde{V}_0^h(\Omega)$, such that

$$a^h(u,v) = f(v) \ \forall \, v \in \ \tilde{V}_0^h(\Omega). \tag{3}$$

3. Nonconforming Coarse Spaces

In this section, we introduce two different types of coarse spaces which make it possible to design efficient domain decomposition methods for problems with discontinuous coefficients in three dimensions.

3.1. A face based coarse space. The first coarse space to be considered, $\tilde{V}_{-1}^F \subset \tilde{V}_0^h(\Omega)$, is based on the average over each face $\mathcal{F}_{ij,h}^{CR}$. Let $\bar{u}_{\mathcal{F}\mathcal{F}_{ij}^{CR}}$ and $\bar{u}_{\partial\Omega_i^{CR}}$ be the average values of u over $\mathcal{F}_{ij,h}^{CR}$ and $\partial\Omega_{i,h}^{CR}$, respectively, and let $\theta_{\mathcal{F}_{ij}}^{CR}$ be the discrete nonconforming harmonic function in Ω_i, in the sense of $a_{\Omega_i}^h(\cdot,\cdot)$, which equals 1 on $\mathcal{F}_{ij,h}^{CR}$ and is zero on $\partial\Omega_{i,h}^{CR} \backslash \mathcal{F}_{ij,h}^{CR}$.

The space $\tilde{V}_{-1}^F$ can conveniently be defined as the range of am interpolation operator $\tilde{I}_h^F : \tilde{V}_0^h(\Omega) \rightarrow \tilde{V}_{-1}^F$, defined by

$$\tilde{I}_h^F u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}^{CR}} \, \theta_{\mathcal{F}_{ij}}^{CR}(x).$$

The associated bilinear form is defined by:

$$b_{-1,F}^{CR}(u,u) = \sum_i \rho_i \{ H(1 + \log H/h) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}^{CR}} - \bar{u}_{\partial\Omega_i^{CR}})^2 \}.$$

3.2. Neumann-Neumann coarse spaces. We consider a family of coarse spaces with only one degree of freedom per substructure; see [13].

For each $\beta \geq 1/2$, we define the pseudo inverses $\mu_{i,\beta}^{+}$, $i = 1, \cdots, N$, by

$$\mu_{i,\beta}^{+}(x) = \frac{1}{(\rho_i)^\beta + (\rho_j)^\beta}, \ x \in \mathcal{F}_{ij,h}^{CR} \ \forall \mathcal{F}_{ij} \subset (\partial\Omega_i \backslash \partial\Omega)$$

and

$$\mu_{i,\beta}^{+}(x) = 0, \ x \in (\Gamma_h^{CR} \backslash \partial\Omega_{i,h}^{CR}) \cup \partial\Omega_h^{CR}.$$

We extend $\mu_{i,\beta}^{+}$ elsewhere in Ω as a nonconforming discrete harmonic function with data on $\Gamma_h^{CR} \cup \partial\Omega_h^{CR}$. The resulting functions belong to $\tilde{V}_0^h(\Omega)$ and are also denoted by $\mu_{i,\beta}^{+}$.

We can now define the coarse space $\tilde{V}_{-1}^{NN} \subset \tilde{V}_0^h(\Omega)$ by

$$\tilde{V}_{-1}^{NN} = \text{span}\{\rho_i^\beta \, \mu_{i,\beta}^{+}\},$$

where the span is taken over all the substructures Ω_i.

We note that $\tilde{V}_{-1}^{NN}$ is also the range of the interpolation operator $\tilde{I}_h^{NN}$ given by

$$u_{-1} = \tilde{I}_h^{NN} u(x) = \sum_i u_{-1}^{(i)} = \sum_i \bar{u}_{\partial\Omega_i^{CR}} (\rho_i)^\beta \mu_{i,\beta}^+. \tag{4}$$

We can even define a Neumann-Neumann coarse space for $\beta = \infty$ [8] by considering the limit of the space $\tilde{V}_{-1}^{NN}$ when β approaches ∞, i.e.

$$\tilde{V}_{-1}^{NN} = \text{span}\{\lim_{\beta\to\infty} \rho_i^\beta \, \mu_{i,\beta}^+\}.$$

The associated bilinear form is defined by:

$$b_{-1,NN}^{CR}(u,u) = a^h(u,u).$$

Lemma 1. *Let $X = F$ or NN. Then, for any $u \in \tilde{V}_0^h(\Omega)$*

$$a^h(\tilde{I}_h^X u, \tilde{I}_h^X u) \le C_3\,(1 + \log H/h)\, a^h(u,u).$$

All constants C_i in this paper are independent of the mesh parameters, β, and the jumps of the coefficients across the interfaces separating the substructures.

4. Multilevel Additive Schwarz Method

Any Schwarz method can be defined by the underlying splitting of the discretization space $\tilde{V}_0^h(\Omega)$ into a sum of subspaces, and by bilinear forms associated with these subspaces. Let $X = F$ or NN. The splitting of $\tilde{V}_0^h(\Omega)$ that we consider is given by

$$\tilde{V}_0^h = \tilde{V}_{-1}^X + \sum_{k=0}^{\ell} \sum_{j\in\mathcal{N}^k} V_j^k + \sum_{j\in\mathcal{N}_h^{CR}} \tilde{V}_j^h.$$

Here, $\mathcal{N}_h^{CR}$ is the set of CR nodal points associated with the space $\tilde{V}_0^h(\Omega)$. The space $\tilde{V}_j^h \subset \tilde{V}_0^h(\Omega)$ is the one-dimensional space spanned by $\tilde{\phi}_j^h$, the standard P_1-nonconforming basis functions associated with the nodes $j \in \mathcal{N}_h^{CR}$. For the definitions of V_j^k and $\mathcal{N}^k$ see [6].

We introduce the following operators:

i) $\tilde{T}_{-1}^X : \tilde{V}_0^h \to \tilde{V}_{-1}^X$, is given by

$$b_{-1,X}^{CR}(\tilde{T}_{-1}^X u, v) = a^h(u,v), \ \forall v \in \tilde{V}_{-1}^X.$$

ii) $\tilde{P}_j^k : \tilde{V}_0^h \to V_j^k$, $k = 0,\cdots,\ell$, $j \in \mathcal{N}^k$ is given by

$$a^h(\tilde{P}_j^k u, v) = a^h(u,v), \ \forall v \in V_j^k.$$

iii) $\tilde{P}_j^h : \tilde{V}_0^h \to \tilde{V}_j^h$, $j \in \mathcal{N}_h^{CR}$, is given by

$$a^h(\tilde{P}_j^h u, v) = a^h(u,v), \ \forall v \in \tilde{V}_j^h.$$

Let

$$\tilde{T}^X = \tilde{T}^X_{-1} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}^k} \tilde{P}^k_j + \sum_{j \in \mathcal{N}^{CR}_h} \tilde{P}^h_j. \tag{5}$$

THEOREM 1. *For any* $u \in \tilde{V}^h_0(\Omega)$

$$C_4 \, (1 + \log H/h)^{-2} a^h(u,u) \le a^h(\tilde{T}^X u, u) \le C_5 \, a^h(u,u).$$

5. A Multiplicative Version

We consider two versions:

$$E_G = (\prod_{j \in \mathcal{N}^{CR}_h} (I - \tilde{P}^h_j))(\prod_{k=0}^{\ell} \prod_{j \in \mathcal{N}^k} (I - \tilde{P}^k_j))(I - \tilde{T}^X_{-1}), \tag{6}$$

and

$$E_J = (I - \eta \sum_{j \in \mathcal{N}^{CR}_h} \tilde{P}^h_j)(\prod_{k=0}^{\ell} (I - \eta \sum_{j \in \mathcal{N}^k} \tilde{P}^k_j))(I - \eta \tilde{T}^X_{-1}), \tag{7}$$

where η is a damping factor such that

$$\|\eta \tilde{T}^X_{-1}\|_{H^1_{\rho,h}}, \; \|\eta \sum_{j \in \mathcal{N}^{CR}_h} \tilde{P}^h_j\|_{H^1_{\rho,h}}, \; \|\eta \sum_{j \in \mathcal{N}^k} \tilde{P}^k_j\|_{H^1_{\rho,h}} \le w < 2.$$

When the product is arranged in an appropriate order, the operators E_G and E_J correspond to the error propagation operator of V-cycle multigrid methods using Gauss Seidel and damped Jacobi smoothers, respectively; see Zhang [**17**]. The norm of the error propagation operators $\|E_G\|_{H^1_{\rho,h}}$ and $\|E_F\|_{H^1_{\rho,h}}$ can be estimated from above by $1 - C_2 \, (\log{(H/h)})^{-2}$.

REMARK 1. *In* [**13**], *we analyzed a two-level additive Schwarz method for discontinuous coefficients. There, we cover* Ω *by overlapping subregions by extending each substructure* Ω_i *to a larger region. We can modify that method by considering inexact local solvers and by covering* Ω *in a different way. The analysis of our methods suggests two attractive ways of covering* Ω*: by face regions* $\Omega_{ij} = \Omega_i \cup \mathcal{F}_{ij} \cup \Omega_j$*, or by cross point regions* Ω'_m *; see* [**6**]. *We again obtain condition number estimates which are polylogarithmically on the number of degree of freedom of individual local subproblems.*

REMARK 2. *We can decrease the complexity of our algorithm by considering approximate discrete nonconforming harmonic extension given by simple explicit formulas in* [**13**].

REMARK 3. *In a case in which we have quasi-monotone coefficients* [**6**] *and use* V^H*, the piecewise linear function, as a coarse space, all algorithms in this paper are optimal.*

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