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A COARSE SPACE FOR ELASTICITY

Partition of Unity Rigid Body Motions Coarse Space

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Abstract In this paper, we develop overlapping Schwarz methods for a finite element discretization on linear elasticity. In order to make Schwarz methods scalable with respect to the number of subdomains, we add an appropriate coarse problem to the algorithms. The main purpose of this paper is to introduce and analyze a new coarse space for these methods. The proposed coarse space is based on a partition of unity (PU) and the local rigid body motions (RM) of the overlapping subdomains. We prove that the condition number of the algorithms grows only linearly with respect to the relative size of the overlap.

Keywords: Schwarz preconditioner, domain decomposition, coarse spaces, partition of unity, elliptic equations, finite elements

1. Introduction

In this paper, we introduce and analyze two-level overlapping Schwarz methods for unstructured meshes for a finite element discretization of linear elasticity. The novel part of the methods is the coarse space. Works on two-level methods and coarse spaces on unstructured meshes is an active area of research. Several different approaches have been introduced and some can be found in [1, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 23] and papers cited therein. Related works to ours, based on two-level agglomeration techniques, can be found in [4, 14]. Their analysis use a class of partition of unity coarse space based on agglomeration smoothing techniques and they have proven an upper bound for the condition number which depends quadratically on the relative overlap. Inspired by some of the numerical results and analysis of [1, 6, 7, 17, 21], the author [19] introduced new coarse basis functions for the discrete Laplacian based on the kind of partition of unity used in the theoretical analysis of Schwarz methods. This kind of partition of unity has a controlled decaying property on the overlapping region and we have proved that the condition number of the algorithms grows only linearly with respect to the relative size of the overlap.

In this paper, we extend some of the results in [19] to linear elasticity. We combine the partition of unity and rigid body motions associated to the (overlapping) subdomains in order to design a new coarse space for elasticity. The partition of unity is used: 1) to localize the coarse basis functions to the subdomains, and 2) to force the coarse basis functions to have a smooth decaying to zero near the boundary of the subdomains. The rigid body motions associated to the subdomains allow us to use some of the theoretical arguments of Neumann-Neumann algorithms [15, 12]. The proposed coarse space, named PU-RM coarse space, has several advantages over traditional coarse spaces: 1) it is applicable to discretizations on unstructured meshes, 2) it is algebraic (see below), 3) the associated algorithms do not require that the subdomains be connected or that the boundary of the subdomains be smooth, 4) the coarse basis functions of the PU-RM coarse space are constructed explicitly and without the use of exact local solvers, 5) the stencil of the coarse matrix is sparser than the traditional ones, and 6) the support of the coarse basis functions is localized on the subdomains, and therefore easy in communication if implemented in a distributed memory parallel machine.

The preconditioners to be considered here are algebraic in the sense that the preconditioners are built in terms of the graph of the sparse matrix and the mesh partition. In this paper we provide a complete and simple mathematical analysis to a finite element problem, the linear elasticity with zero displacement (Dirichlet) boundary condition.

2. Linear Elasticity Discretization

We consider an isotropic elastic material in the configuration region $\Omega \subset \Re^2$. Let us denote $u^* = (u_1^*, u_2^*)^t$ to be the displacement and $f = (f_1, f_2)^t$ the body force. Let the region Σ be Ω or a subregion of Ω, and the spaces $\vec{H}^1(\Sigma)$, $\vec{L}_2(\Sigma)$, and $\vec{H}_0^1(\Sigma)$ to be the spaces $(H^1(\Sigma), H^1(\Sigma))^t$, $(L_2(\Sigma), L_2(\Sigma))^t$, and $(H_0^1(\Sigma), H_0^1(\Sigma))^t$, respectively. The weak formulation of the static theory of linear elasticity with zero boundary displacement boundary condition is given as follows.

Find $u^* \in \vec{H}_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \ \ \forall v \in \vec{H}_0^1(\Omega), \tag{1}$$

where

$$a(u^*, v) = \int_\Omega \left(\mu E(u^*) : E(v) + \lambda \mathrm{div}(u^*)\ \mathrm{div}(v)\right)\ dx\ ,$$

$$f(v) = \int_\Omega f \cdot v \, dx \ \text{ for } \ f \in \vec{L}^2(\Omega),$$

and

$$E(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^t\right).$$

The positive constants μ and λ are called the Lamé constants. It is well-known that $a(\cdot,\cdot)$ is elliptic (using the First Korn Inequality; see Theorem 9.2.25 in [3]) and bounded, and therefore the system (1) is well posed.

For simplicity, let Ω be a bounded polygonal region in $\Re^2$ with a diameter of size $O(1)$. The extension of the results to $\Re^3$ can also be carried out using similar ideas. Let $T^h(\Omega)$ be a shape regular, quasi-uniform triangulation of grid size $O(h)$ of Ω, and $V \subset \vec{H}^1_0(\Omega)$ be the finite element space consisting of continuous piecewise linear functions associated with the triangulation. The extension of the theory for the case of local quasi-uniform triangulation is also straightforward.

We are interested in solving the discrete problem associated to (1): Find $u \in V$ such that

$$a(u,v) = f(v), \ \ \forall\, v \in V. \tag{2}$$

Since $V \subset \vec{H}^1_0(\Omega)$, the discrete version is also well-posed.

Throughout this paper, C and C_0, are positive generic constants that do not depend of any of the mesh parameters, the number of subdomains, and the parameters λ and μ. All the domains and subdomains are assumed to be open; i.e., boundaries are not included in their definitions.

3. Algebraic Subregions

Given the domain Ω and triangulation $T^h(\Omega)$, we assume that a domain partition has been applied and resulted in N substructures (non-overlapping subregions) $\Omega_i, i = 1, \ldots N$, of size $O(H)$, such that

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for} \quad j \neq i.$$

We define the overlapping subdomains Ω_i^δ as follows. Let Ω_i^1 be the one-overlap element extension of Ω_i, where $\Omega_i^1 \supset \Omega_i$ is obtained by including all the immediate neighboring elements $\tau_h \in T^h(\Omega)$ of Ω_i such that $\overline{\tau}_h \cap \overline{\Omega}_i \neq \emptyset$. Using the idea recursively, we can define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i \subset \Omega_i^1 \subset \cdots \Omega_i^\delta.$$

Here the integer $\delta \geq 1$ indicates the level of element extension and δh is the approximate length of the extension. We note that this extension can be coded easily through the knowledge of the adjacent matrix associated to the mesh and the partition.

In this paper we consider the linear elasticity problem for the case of Dirichlet boundary condition on the whole $\partial\Omega$. We want to design coarse spaces based on a class of partition of unit for which has been used as a very powerful tool in the theoretical analysis of Schwarz type domain decomposition methods. We note however that the partition of unity functions on this class do not necessarily vanish on $\partial\Omega$. Hence, they cannot be used straightforwardly as coarse basis functions since they should satisfy zero Dirichlet boundary conditions for Dirichlet type boundary problems. Hence, for the coarse basis functions that touch $\partial\Omega$, we modify them so that they have a controlled decaying to zero near $\partial\Omega$. To obtain such coarse basis functions, we next introduce a Dirichlet boundary treatment. Let Ω_B^1 be one layer of elements near the Dirichlet boundary $\partial\Omega$ and then define recursively,

$$\Omega_B^1 \subset \Omega_B^2 \cdots \Omega_B^\delta$$

with δ levels of extension by adding recursively neighboring elements.

To define and analyze the new methods, we subdivide Ω_i^δ as follow. Let $\gamma_i^\delta = \partial\Omega_i^\delta \backslash \partial\Omega, i = 1, \cdots, N$; i.e., the part of the boundary of Ω_i^δ that does not belong to the physical boundary of Ω, and let $\gamma_B^\delta = \partial\Omega_B^\delta \backslash \partial\Omega$. We define the interface overlapping boundary Γ^δ as the union of all the γ_i^δ and γ_B^δ; i.e., $\Gamma^\delta = \cup_{i=B,1}^N \gamma_i^\delta$. We also need the following subsets of Ω_i^δ

- $\Gamma_i^\delta = \Gamma^\delta \cap \Omega_i^\delta$ (local interface)
- $N_i^\delta = \Omega_i^\delta \backslash (\cup_{j\neq i}\Omega_j^\delta \cup \Omega_B^\delta \cup \Gamma_i^\delta)$ (non-overlapping region)
- $O_i^\delta = \Omega_i^\delta \backslash (N_i^\delta \cup \Gamma_i^\delta)$ (overlapping region)

We note that $\Omega_i^\delta = N_i^\delta \cup \Gamma_i^\delta \cup O_i^\delta$. The region O_i^δ is the overlapping region of Ω_i^δ excluding $\Gamma_i^\delta \cap \Omega_i^\delta$. The region N_i^δ is the subregion of Ω_i^δ which does not overlap any neighboring extended subdomain $\overline{\Omega}_j^\delta$ and $\overline{\Omega}_B^\delta$. We recall that the regions O_i^δ and N_i^δ are open sets.

4. Overlapping Schwarz Methods

We next describe the PU-RM coarse space and introduce the corresponding overlapping additive Schwarz method and a hybrid Schwarz method. We first consider the local spaces and local problems.

4.1 The Local Spaces

We introduce the local spaces as

$$V_i^\delta = V \cap \vec{H}_0^1(\Omega_i^\delta), \quad i = 1, 2, \cdots, N,$$

extended by zero to $\Omega\backslash\Omega_i^\delta$. It is easy to verify that

$$V = V_1^\delta + V_2^\delta + \cdots + V_N^\delta. \quad (3)$$

The property (3) gives robustness for the preconditioners defined in this paper in the sense that a convergence is always attained independently of the quality of the partitioning. This is an advantage over some iterative substructuring methods in which are based on the requirement that all the substructures Ω_i must be connected. We point out that the space decomposition given by (3) is not a direct sum if $\delta > 1$. This increases robustness for the methods when the Ω_i have rough (*zigzag*) boundaries. By extending the substructures to Ω_i^δ, we allow the possibility of decomposing a function of V as a sum of functions of V_i^δ without the *zigzag* behavior. So it is possible to obtain low energy decompositions, and hence better lower bounds for the condition number of the preconditioners are obtained.

We define the local projections (local problems) $P_i^\delta : V \to V_i^\delta$ as follows:

$$a(P_i^\delta u, v) = a(u, v), \quad \forall v \in V_i^\delta. \quad (4)$$

We next introduce the PU-RM coarse space V_0^δ.

4.2 A PU-RM Coarse Space

We next construct a partition of unity θ_i^δ such that $\theta_i^\delta \in V_i^\delta$, $0 \le \theta_i^\delta(x) \le 1$, $|\nabla\theta_i^\delta(x)| \le C/(\delta h)$ in the interior of the elements, and $\sum_{i=B,1}^N \theta_i^\delta \equiv 1$. Such construction is natural, algebraic and easy to implement. We first construct the function $\hat\theta_B^\delta \in V_i^\delta$ as follows. We let $\hat\theta_B^\delta(x) = 1$ for nodes x on $\partial\Omega$. For the first layer of neighboring nodes x of $\partial\Omega$ we let $\hat\theta_B^\delta(x) = (\delta-1)/\delta$, For the second layer of neighboring nodes x of $\partial\Omega$ we let $\hat\theta_B^\delta(x) = (\delta-2)/\delta$, and recursively until $k = \delta - 1$, we let $\hat\theta_B^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\partial\Omega$. For the reminaing nodes x of $\overline{\Omega}$ we let $\hat\theta_B^\delta(x) = 0$. Similarly, for $i = 1, \cdots, N$, we let $\hat\theta_i^\delta(x) = 1$ for nodes x of $\overline{\Omega}_i\backslash\Omega_B^\delta$. For the first layer of neighboring nodes x of $\overline{\Omega}_i\backslash\Omega_B^\delta$ we let $\hat\theta_i^\delta(x) = (\delta-1)/\delta$, and recursively until $k = \delta - 1$, we let $\hat\theta_i^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\overline{\Omega}_i\backslash\Omega_B^\delta$. For the remaining nodes x of $\overline{\Omega}$ we let $\hat\theta_i^\delta(x) = 0$. It is easy to verify that $0 \le \hat\theta_i^\delta(x) \le 1$, and for quasi-uniform triangulation $|\nabla\hat\theta_i^\delta(x)| \le C/(\delta h)$ in the interior of the elements. The partition of unity θ_i^δ is defined as

$$\theta_i^\delta = I_h\left(\frac{\hat\theta_i^\delta}{\sum_{j=B,1}^N \hat\theta_j^\delta}\right).$$

Here I_h is the regular pointwise linear interpolation operator from the continuous functions to piecewise linear and continuous functions. It is easy to verify that $\sum_{i=B,1}^N \theta_i^\delta(x) = 1,\ \ 0 \le \theta_i^\delta(x) \le 1$, and $|\nabla\theta_i^\delta(x)| \le C/(\delta h), \forall x \in \overline{\Omega}$.

We next consider the key ingredient for designing the coarse space for linear elasticity: the local rigid body motions. We let

$$R\vec{M}(\Sigma) = \{v \in \vec{L}_2(\Sigma) : v = c + b(x_2, -x_1)^t, c \in \Re^2, b \in \Re\}$$

be the space of rigid body motions functions on Σ. An important property of the space $R\vec{M}(\Sigma)$, and which will play an important role in the later analysis of our methods, is given as follows. If $v \in R\vec{M}(\Sigma)$ then $a_\Sigma(v,v) \equiv 0$. In addition, in certain extent the converse is also true; if $a_\Sigma(v,v) \equiv 0$ and Σ is connected, then $v \in R\vec{M}(\Sigma)$. Here, the bilinear form $a_\Sigma(\cdot,\cdot)$ on $\vec{H}^1(\Sigma) \times \vec{H}^1(\Sigma)$ is given by

$$a_\Sigma(u,v) = \int_\Sigma (2\mu E(u) : E(v) + \lambda \text{div}(u)\,\text{div}(v))\ dx\ .$$

The PU-RM coarse space V_0^δ is defined as

$$V_0^\delta = \{\sum_{i=1}^N \vec{I}_h \left([c_i + b_i(x_2, -x_1)^t]\theta_i^\delta\right), \forall c_i \in \Re^2, \forall b_i \in \Re\}. \tag{5}$$

Here, the interpolator $\vec{I}_h = (I_h, I_h)^t$ is the regular componentwise pointwise linear interpolation operator. We note that we do not include the θ_B^δ in the sum of (5), and therefore the number of degrees of freedom of V_0^δ is $3N$. The function θ_B^δ is needed only to define the others functions θ_i^δ, $i = 1, \cdots, N$.

We define the global projection (global problem) $P_0^\delta : V \to V_0^\delta$ as follows:

$$a(P_0^\delta u, v) = a(u,v), \quad \forall v \in V_0^\delta.$$

4.3 Preconditioners

We consider two preconditioners:

- The two-level overlapping additive Schwarz operator [9] given by

$$P_{as}^\delta = \sum_{i=0}^N P_i^\delta,$$

- The hybrid Schwarz operator [15, 16] given by

$$P_{hyb}^\delta = P_0^\delta + (I - P_0^\delta)(\sum_{i=1}^N P_i^\delta)(I - P_0^\delta).$$

5. Theoretical Analysis

It is possible to show that the solution of (2) is the solution of the preconditioned system $P_{as}^{\delta} u = g_{as}$ ($P_{hyb}^{\delta} u = g_{hyb}$), for an appropriate right hand side g_{as} (g_{hyb}); see [20]. These preconditioned systems are typically solved by the conjugate gradient method, without further preconditioning, using $a(\cdot,\cdot)$ as the inner product. The preconditioned systems presented in this paper are applicable to any unstructure mesh and partitioning. The notions of subdomains, the classification of the regions O_i^{δ} and N_i^{δ} and the interfaces Γ_i^{δ}, etc., can all be defined in terms of the graph of the sparse matrix. In this section we provide a sharp and optimal estimate for a case in which each Ω_i^{δ} has a square shape or is an image of a reference square under reasonably smooth mapping. The analysis below can then be carried out on the reference square. We note however, that results can also be extended for any connected Lipschitz subregions that are not too distorted. We next prove the main result of the paper.

Theorem 1. *There exists a constant $C > 0$ such that for any $u \in V$ we have*

$$\kappa(P_{hyb}^{\delta}) \leq \kappa(P_{as}^{\delta}) \leq C(1 + \lambda/\mu)(1 + \frac{H}{\delta h}). \tag{6}$$

The constant C does not depend on h, δ, H, λ, and μ.

Proof.

We estimate the condition numbers of the operators P_{as}^{δ} and P_{hyb}^{δ} in terms of the fine mesh size h, the subdomain size H, the overlapping factor δ, and the parameters λ and μ. We shall follow the abstract additive Schwarz theory [22, 20] to analyze the additive versions, where three assumptions have to be checked and three paramenters C_0, ω and $\rho(E)$ estimated; see Theorem 1 in [20]. Two assumptions are trivial to check: $\omega = 1$ since we use exact solvers, and $\rho(E) \leq C$, where C is the maximum number of subdomains Ω_i^{δ} overlapping a common point. Hence, if the overlap is of the order of H or less, then C will not depend of H and h. So our focus on the rest of the paper is in bounding C_0. The first inequality of (6) follows directly from the Lemma 3.2 in [16]; see also Theorem 4 in [20].

What remains to complete the proof is to derive a bound for C_0; i.e., to find C_0 such that for any given $u \in V$, there exist $u_i \in V_i^{\delta}$, such that

$$u = \sum_{i=0}^{N} u_i, \tag{7}$$

and

$$\sum_{i=0}^{N} a(u_i, u_i) \leq C_0^2 a(u, u). \tag{8}$$

We next concentrate on defining the decomposition $u = \sum_{i=0}^{N} u_i$. Let $V_{i,0}^{\delta}, i = 1, \cdots, N$, be the spaces with three degrees of freedom generated by the $[c_i^{\delta} + b_i^{\delta}(x_2, -x_1)^t]\theta_i^{\delta}$ coarse basis functions. We introduce the interpolation-like operator $I_0^{\delta} = \sum_{i=1}^{N} I_{i,0}^{\delta}$, where the $I_{i,0}^{\delta} : V \rightarrow V_{i,0}^{\delta}$ is defined as follows:

$$(I_{i,0}^{\delta}u)(x) := \vec{I}_h \left([\bar{c}_i^{\delta} + \bar{b}_i^{\delta}(x_2, -x_1)^t]\theta_i^{\delta}(x) \right),$$

where

$$\bar{b}_i^{\delta} = -\frac{1}{2|\Omega_i^{\delta}|} \int_{\Omega_i^{\delta}} \operatorname{rot}(u(x))\, dx$$

and

$$\bar{c}_i^{\delta} := \frac{1}{|\Omega_i^{\delta}|} \int_{\Omega_i^{\delta}} [u(x) - \bar{b}_i^{\delta}(x_2, -x_1)^t]\, dx.$$

Here $|\Omega_i^{\delta}|$ is the area of the region Ω_i^{δ}. Let us denote

$$w_i = u - [\bar{c}_i^{\delta} + \bar{b}_i^{\delta}(x_2, -x_1)^t].$$

We remark that the definitions of $\bar{b}_i^{\delta}$ and the $\bar{c}_i^{\delta}$ imply that $\operatorname{rot}(w_i)$ and the two components of w_i have average zero on Ω_i^{δ}. We define the u_i as follows. Let $u_0 \in V_0^{\delta}$ be defined as

$$u_0 = I_0^{\delta}u = \sum_{i=1}^{N} I_0^{i,\delta}u = \sum_{i=1}^{N} \vec{I}_h \left([\bar{c}_i^{\delta} + \bar{b}_i^{\delta}(x_2, -x_1)^t]\theta_i^{\delta} \right),$$

and the $u_i \in V_i^{\delta}, i = 1, \cdots, N$ as

$$u_i = \vec{I}_h(\vartheta_i^{\delta}u) - \vec{I}_h \left([\bar{c}_i^{\delta} + \bar{b}_i^{\delta}(x_2, -x_1)^t]\theta_i^{\delta} \right).$$

Here, the piecewise linear functions $\vartheta_i^{\delta} \in H^1(\Omega_i^{\delta})$ are modifications of the θ_i^{δ} and are defined so that they form a partition of unity $\sum_{i=1}^{N} \vartheta_i^{\delta} \equiv 1$ on the whole $\overline{\Omega}$, satisfy $0 \le \vartheta_i^{\delta} \le 1$, $|\nabla \vartheta_i^{\delta}| \le C/(\delta h)$, and $\vartheta_i^{\delta}(x) = \theta_i^{\delta}(x), \forall x \in \Omega \backslash \overline{\Omega}_B^{\delta}$. It is therefore easy to see that (7) holds. We postpone the construction of the ϑ_i^{δ} to later in this paper. We note that the ϑ_i^{δ} are not used in the algorithms; they are used only as a tool for the analysis.

The strategy of the proof is to first obtain a bound for $\sum_{i=1}^{N} a(u_i, u_i)$. Once done, we obtain a bound for $\sum_{i=0}^{N} a(u_i, u_i)$ by noticing that $u_0 = u - \sum_{i=1}^{N} u_i$, and therefore with triangular inequalities and properties of $\rho(\mathcal{E})$ we obtain

$$a(u_0, u_0) \le 2 \left(a(u, u) + a(\sum_{i=1}^{N} u_i, \sum_{i=1}^{N} u_i) \right)$$

$$\leq 2\left(a(u,u)+\rho(\mathcal{E})\sum_{i=1}^{N}a(u_i,u_i)\right).$$

It remains to obtain a bound for $\sum_{i=1}^{N} a(u_i,u_i)$. To do so, we first decompose the $u_i, i=1,\cdots,N$ as $u_i = u_i^0 + u_i^B$ where

$$u_i^0 = \vec{I}_h(\theta_i^\delta w_i) \text{ and } \quad u_i^B = \vec{I}_h((\vartheta_i^\delta - \theta_i^\delta)u)), \tag{9}$$

and use

$$\sum_{i=1}^{N} a(u_i,u_i) \leq 2\sum_{i=1}^{N} a(u_i^0,u_i^0) + 2\sum_{i=1}^{N} a(u_i^B,u_i^B), i=1,\cdots,N.$$

The next steps concentrate in obtaining a bound for each $a(u_i^0,u_i^0)$. The estimation of $a(u_i^0,u_i^0)$ is done as in standard additive Schwarz methods [9]. Let K be an element of Ω_i^δ, and $\bar{\theta}_i^{K,\delta}$ be the average of θ_i^δ on K. We have

$$a_K(u_i^0,u_i^0) = a_K(\vec{I}_h(\theta_i^\delta w_i), \vec{I}_h(\theta_i^\delta w_i))$$

$$\leq 2a_K(\vec{I}_h(\bar{\theta}_i^{K,\delta} w_i), \vec{I}_h(\bar{\theta}_i^{K,\delta} w_i)) \tag{10}$$

$$+ 2a_K(\vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i), \vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i)). \tag{11}$$

For the term (10), we use that $|\bar{\theta}_i^{K,\delta}| \leq 1$ (it follows from $0 \leq \theta_i^\delta \leq 1$) to obtain

$$a_K(\vec{I}_h(\bar{\theta}_i^{K,\delta} w_i), \vec{I}_h(\bar{\theta}_i^{K,\delta} w_i)) \leq a_K(w_i,w_i).$$

For obtaining a bound for the term (11) we do the following. If $K \subset N_i^\delta$ then $\theta_i^\delta \equiv 1$, and hence $\|\theta_i^\delta - \bar{\theta}_i^{K,\delta}\|_{L^\infty(K)} = 0$ and (11) vanishes. It remains to consider only the case $K \subset O_i^\delta$. To do so, we first use an inverse inequality; i.e., any linear function v defined on an element K of size $O(h)$ satisfies $|v|_{H^1(K)} \leq C/h\|v\|_{L^2(K)}$. We obtain

$$a_K(\vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i), \vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i))$$

$$\leq C\max\{\mu,\lambda\}h^{-2}\|(\vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i)\|^2_{\vec{L}^2(K)}.$$

We then use that $\|\theta_i^\delta - \bar{\theta}_i^{K,\delta}\|_{L^\infty(K)} \leq C/\delta$ (it follows from $|\nabla\theta_i^\delta| \leq C/(\delta h)$) to have

$$C\max\{\mu,\lambda\}h^{-2}\|(\vec{I}_h([\theta_i^\delta - \bar{\theta}_i^{K,\delta}]w_i)\|^2_{\vec{L}^2(K)}$$

$$\leq C\max\{\mu,\lambda\}(\delta h)^{-2}\|w_i\|^2_{\vec{L}^2(K)}.$$

Hence, using that the support u_i^0 is contained in $\overline{\Omega}_i^\delta$, and the previous calculations, we have

$$a(u_i^0, u_i^0) = \sum_{K \in \Omega_i^\delta} a_K(u_i^0, u_i^0)$$

$$\leq C \left(a_{\Omega_i^\delta}(w_i, w_i) + \max\{\mu, \lambda\} \frac{1}{(\delta h)^2} \|w_i\|^2_{\vec{L}^2(O_i^\delta)} \right).$$

We next use similar arguments as in Lemma 3 in [11]. By using simple manipulations of the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality, we have

$$\|w_i\|^2_{\vec{L}_2(O_i^\delta)} \leq C \left(\delta h \|w_i\|^2_{\vec{L}_2(\partial \Omega_i^\delta)} + (\delta h)^2 |w_i|^2_{\vec{H}^1(O_i^\delta)} \right),$$

and with a trace theorem for domains (here Ω_i^δ) with size of $O(H)$ we obtain

$$\|w_i\|^2_{\vec{L}_2(\partial \Omega_i^\delta)} \leq C \left(H^{-1} \|w_i\|^2_{\vec{L}^2(\Omega_i^\delta)} + H |w_i|^2_{\vec{H}^1(\Omega_i^\delta)} \right).$$

Hence,

$$(\delta h)^{-2} \|w_i\|^2_{\vec{L}^2(O_i^\delta)} \leq C \left((1 + \frac{H}{\delta h}) |w_i|^2_{\vec{H}^1(\Omega_i^\delta)} + \frac{1}{H(\delta h)} \|w_i\|^2_{\vec{L}^2(\Omega_i^\delta)} \right),$$

and therefore,

$$a(u_i^0, u_i^0) \leq C \max\{\mu, \lambda\} \left((1 + \frac{H}{\delta h}) |w_i|^2_{\vec{H}^1(\Omega_i^\delta)} + \frac{1}{H(\delta h)} \|w_i\|^2_{\vec{L}^2(\Omega_i^\delta)} \right).$$

Using that $\mathrm{rot}(w_i)$ and the two components of w_i have average zero on Ω_i^δ, we can apply the Second Korn inequality (see Theorem 9.2.12 in [3]) to obtain

$$|w_i|^2_{\vec{H}^1(\Omega_i^\delta)} + \frac{1}{H^2} \|w_i\|^2_{\vec{L}_2(\Omega_i^\delta)} \leq C \int_{\Omega_i^\delta} E(w_i) : E(w_i) \, dx,$$

and since

$$\int_{\Omega_i^\delta} \mathrm{div}(w_i) \, \mathrm{div}(w_i) \, dx \ \leq C \int_{\Omega_i^\delta} E(w_i) : E(w_i) \, dx,$$

we obtain

$$a(u_i^0, u_i^0) \leq C(1 + \lambda/\mu)(1 + \frac{H}{\delta h}) a_{\Omega_i^\delta}(w_i, w_i).$$

We then use that $u - w_i \in \vec{RM}(\Omega_i^\delta)$ (kernel of $a_{\Omega_i^\delta}(\cdot, \cdot)$) to obtain

$$a_{\Omega_i^\delta}(w_i, w_i) = a_{\Omega_i^\delta}(u, u).$$

The next step is to obtain a bound for $a(u_i^B, u_i^B)$. We note that for $i = 1, \cdots, N$

$$\vartheta_i^\delta(x) = \theta_i^\delta(x), \forall x \in \Omega \backslash \overline{\Omega}_B^\delta,$$

and therefore the support of u_i^B is on $\overline{\Omega}_B^\delta \cap \overline{\Omega}_i^\delta$. On $\overline{\Omega}_B^\delta$, we have $|\vartheta_i^\delta - \theta_i^\delta| \leq 1$ and $|\nabla \vartheta_i^\delta - \nabla \theta_i^\delta| \leq C/(\delta h)$. So we can use the similar arguments as before and a Friedrichs inequality (u vanishes on $\partial\Omega$) to obtain,

$$a(u_i^B, u_i^B) \leq C(1 + \lambda/\mu)(1 + \frac{H}{\delta h}) a_{\Omega_i^\delta}(u, u).$$

We now sum all the contributions and use a coloring argument to obtain

$$\sum_{i=1}^{N} a(u_i, u_i) \leq C(1 + \lambda/\mu) a(u, u).$$

We now give the construction of the ϑ_i^δ. We next modify the coarse basis functions θ_i^δ on $(\overline{\Omega}_i^\delta \cap \overline{\Omega}_B^\delta)$ to define the partition of unity ϑ_i^δ. We first construct the function $\hat{\vartheta}_i^\delta \in H^1(\Omega_i^\delta)$. Let $\hat{\vartheta}_i^\delta(x) = 1$ and $\hat{\vartheta}_i^\delta(x) = 0$ for nodes x of $\overline{\Omega}_i$ and $\overline{\Omega} \backslash \Omega_i^\delta$, respectively. For the first layer of neighboring nodes x of $\overline{\Omega}_i$ we let $\hat{\vartheta}_i^\delta(x) = (\delta - 1)/\delta$, and recursively until $k = \delta - 1$, we let $\hat{\vartheta}_i^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\overline{\Omega}_i$. The partition of unity ϑ_i^δ is defined as

$$\vartheta_i^\delta = I_h\left(\frac{\hat{\vartheta}_i^\delta}{\sum_{j=1}^{N} \hat{\vartheta}_j^\delta}\right).$$

It is easy to verify that $\sum_{i=1}^{N} \vartheta_i^\delta(x) = 1, \quad 0 \leq \vartheta_i^\delta(x) \leq 1$, and $|\nabla \vartheta_i^\delta(x)| \leq C/(\delta h)$ in the interior of the elements. Also, $\vartheta_i^\delta(x) = \theta_i^\delta(x), i = 1, \cdots, N$, when $x \in \Omega \backslash \Omega_B^\delta$.

□

We note that the discretization considered in this paper gives satisfactory (second order accurate) convergent finite element approximation to the elasticity problem when λ/μ is not large. It can be shown [2, 3] that the apriori error estimate of this finite element method deteriorates as $\lambda \gg \mu$; this phenomenon is called *locking effect* or *volume locking*. We note that the upper bound estimate of the preconditioners presented here also follows the similar patterns. Here also, we cannot remove the λ/μ dependence on the upper bound estimates for the conditioning number of the preconditioned systems. To see this we use the following arguments. If $\text{div}(u) = 0$ and λ is close to ∞, the only way to obtain a decomposition stable with respect to λ is to have the all the $\text{div}(u_i) = 0$. However, it is easy to see that $\text{div}(u_0) = 0$ implies that u_0 vanishes. Hence, there is no global communication and therefore the condition number must have a H dependence on the upper bound estimation. For incompressible ($\lambda = \infty$) or almost incompressible materials, other discretizations based on hybrid or non-conforming finite elements approximations [2, 3] are more appropriate and they will not be considered here.

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