NEUMANN-NEUMANN METHODS FOR A DG DISCRETIZATION ON GEOMETRICALLY NONCONFORMING SUBSTRUCTURES

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ABSTRACT. A discontinuous Galerkin discretization for second order elliptic equations with *discontinuous coefficients* in 2-D is considered. The domain of interest Ω is assumed to be a union of polygonal substructures Ω_i of size $O(H_i)$. We allow this substructure decomposition to be geometrically nonconforming. Inside each substructure Ω_i, a conforming finite element space associated to a triangulation $\mathcal{T}_{h_i}(\Omega_i)$ is introduced. To handle the nonmatching meshes across $\partial\Omega_i$, a discontinuous Galerkin discretization is considered. In this paper additive and hybrid Neumann-Neumann Schwarz methods are designed and analyzed. Under natural assumptions on the coefficients and on the mesh sizes across $\partial\Omega_i$, a condition number estimate $C(1+\max_i \log \frac{H_i}{h_i})^2$ is established with C independent of h_i, H_i, h_i/h_j, and the jumps of the coefficients. The method is well suited for parallel computations and can be straightforwardly extended to three dimensional problems. Numerical results are included.

1. INTRODUCTION

In this paper we consider a *discontinuous Galerkin* (DG) approximation of elliptic problems with *discontinuous coefficients*. More precisely, we consider the problem

$$\left\{ \begin{array}{rll} -\mathrm{div}(\rho(x)\nabla u) & = f & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is a two-dimensional polygonal region and the coefficient ρ might be a discontinuous function. We use the DG discretization introduced in [9] and described below. For an overview on local DG discretizations we refer to [4, 30] and references therein. The domain Ω is a *geometrically nonconforming* union of disjoint polygonal substructures Ω_i, $i = 1, \ldots, N$. Large discontinuities of the coefficients are assumed to occur only across the interfaces between substructures. For simplicity of presentation we assume that inside each substructure Ω_i the coefficient is a positive constant, i.e, $\rho(x) = \rho_i > 0$ for all $x \in \Omega_i$. The extension of the results to mildly variation of ρ inside Ω_i is straightforward. Inside each substructure Ω_i a conforming finite element method is introduced to discretize the local problem. *Nonmatching triangulations* are allowed to occur across the substructures boundaries $\partial\Omega_i$, $i = 1, \ldots, N$. This kind of composite discretization is motivated by the location of the discontinuities of the coefficient $\rho(x)$ and by the regularity of the solution of the problem. The discrete problem is formulated using a symmetric DG

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method with interior penalty (IPDG) terms on $\partial\Omega_i$. To deal with the discontinuities of the coefficients across the substructure interfaces, *harmonic averages* of the values of the coefficient are considered in the bilinear forms associated to substructure boundaries in the IPDG formulation; see [9] and Section 2 below. The consistency of this discretization is given in [10]. An optimal a priori error estimate is established in [9]; see also Lemma 2.4 below. IPDG methods based on harmonic averages of the coefficients were also considered for advection-diffusion-reaction problems to obtain stable discretizations; see [7].

The main goal of this paper is to design and analyze additive and hybrid Neumann-Neumann (N-N) algorithms for the resulting DG-discrete problem. This type of algorithms are well established for the solution of discrete problems resulting from standard conforming and nonconforming discretizations; see [14, 28, 29, 33, 32, 22, 19, 20]. However, not enough attention has been payed to DG discretizations. We note that other types of preconditioners have been considered for solving discrete IPDG problems. In connection with two-level domain decomposition preconditioners, we mention [15, 16, 25, 6, 1, 2, 8, 27], where small and generous overlapping Schwarz methods were considered for DG discretizations. In connection with multilevel preconditioners for DG problems, we mention [17, 21, 26, 24, 23, 5]. These papers focus on the scalability of the preconditioners with respect to mesh parameters, however, little has been said about the robustness with respect to jumps of the coefficient and nonmatching grids across the substructuring interfaces. It is known that, in the case of classical conforming and nonconforming discretizations in two dimensions, domain decomposition and multilevel methods may lead to robust preconditioners with respect to jumps of the coefficient; see [33]. In three dimensions, however, the robustness of these methods with respect to the jumps in the coefficient can be achieved only in special circumstances such as when every substructure touches part of the Dirichlet boundary or when only few cross points do not satisfy the quasi-monotonicity condition on the jumps of the coefficient; see [12, 34, 18]. For discontinuous coefficients, the robustness of these methods can be achieved when coarse problems based on discrete harmonic extensions are introduced; see [12, 14, 29, 32, 13, 31]. The same robustness issues also occur for DG discretizations and motivated the introduction of the notion of discrete harmonic extension in the DG sense in [10, 11]. In [10, 11] the authors consider discontinuous coefficients and the case where Ω is a *geometrically conforming* union of substructures Ω_i, $i = 1, \ldots, N$. In this paper we consider discontinuous coefficients and the *geometrically nonconforming* case. We design and analyze several DG Neumann-Neumann coarse spaces and solvers for the case where the domain Ω is the nonconforming union of substructures Ω_i, $i = 1, \ldots, N$.

As a first step toward our developments, the original DG discrete problem is reduced to the Schur complement form with respect to unknowns associated to the substructures boundaries $\partial\Omega_i$, $i = 1, \ldots, N$. Discrete harmonic functions defined in a special way are required in this step; see Section 3. After a Schur complement problem is posed, we use the general theory of N-N methods (see e.g. [33]) to design and analyze our methods. At this step, special local problems and special coarse problems are designed. The local problem associated to the substructure Ω_i includes all the Schur complement degrees of freedom associated to $\partial\Omega_i$. Due to the nature of the DG discretization we are considering, this local problem actually includes degrees of freedom within the substructure Ω_i and also includes degrees of freedom within the neighbor subdomains Ω_j that are associated to the edges or part of edges of $\partial\Omega_j$ which are common to Ω_i. The coarse space is defined by using a special partitioning of unity with respect to the substructures Ω_i and by introducing master and slave sides of the local interfaces between the substructures. We prove that the algorithms are almost optimal and their rates of convergence are independent of the mesh parameters, the number of substructures Ω_i and the jumps of the coefficient. The algorithms are well suited for parallel computations and they can be straightforwardly extended to three-dimensional problems. Despite of the fact the extension of our results to the 3-D case is easy, it needs a lot of details and therefore is not discussed in the paper. It will be considered separately.

The paper is organized as follows. In Section 2 the differential problem and its DG discretization are formulated. In Section 3 the Schur complement problem is derived using discrete harmonic functions in a special way. Section 4 is dedicated to introduce notation and the *interface condition* on the coefficient and the mesh parameters, see Assumption 4.1. Two additive N-N Schwarz preconditioners, one based on a small coarse space and another based on a larger coarse space, are defined and analyzed in Section 5. In Section 6 we present the Balancing Domain Decomposition preconditioning versions. Finally, in Section 7 some numerical experiments are presented which confirm the theoretical results. The numerical results show that the introduced Assumption 4.1 is necessary and sufficient to obtain robustness with respect to the jump of the coefficient and the mesh size ratios.

2. Differential and discrete problems

We consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \quad \text{for all } v \in H_0^1(\Omega) \tag{2.1}$$

where

$$a(u,v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \text{ and } f(v) := \int_{\Omega} f v dx.$$

Here, $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ where the substructures $\{\Omega_i\}_{i=1}^N$ are disjoint regular polygonal subregions of diameter $O(H_i)$. We assume that the substructures $\{\Omega_i\}_{i=1}^N$ form a geometrically nonconforming partition of Ω. In this case, for $i \neq j$, the intersection $\partial\Omega_i \cap \partial\Omega_j$ is either empty, a vertex of Ω_i and/or Ω_j, or a common edge or part of an edge of $\partial\Omega_i$ and $\partial\Omega_j$. We recall that in the case of geometrically conforming decomposition, the intersection $\partial\Omega_i \cap \partial\Omega_j$ is either empty or a common vertex of Ω_i and Ω_j, or a common edge of Ω_i and Ω_j. For simplicity of presentation we assume that the right-hand side $f \in L^2(\Omega)$ and the value of the coefficient ρ_i on Ω_i is a positive constant, $i = 1, \ldots, N$.

In each Ω_i we introduce a shape regular triangulation $\mathcal{T}_{h_i}(\Omega_i)$ with triangular elements and mesh parameter h_i. The resulting triangulation of Ω is in general nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be the regular finite element space of piecewise linear and continuous functions in $\mathcal{T}_{h_i}(\Omega_i)$. We do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. We define

$$X_h(\Omega) = X_1(\Omega_1) \times \cdots \times X_N(\Omega_N)$$

and represent functions v of $X_h(\Omega)$ as $v = \{v_i\}_{i=1}^N$ with $v_i \in X_i(\Omega_i)$.

Remark 2.1 (Notation). Let $E = \partial\Omega_i \cap \partial\Omega_j$. Due to the fact that $\mathcal{T}_{h_i}(\Omega_i)$ and $\mathcal{T}_{h_j}(\Omega_j)$ are independent from each other, it is useful to distinguish between $E \subset \overline{\Omega}_i$

and $E \subset \overline{\Omega}_j$. From now on we will distinguish between the Ω_i-side of E, denoted by E_{ij} and the Ω_j-side of E, denoted by E_{ji}. Geometrically, E_{ij} and E_{ji} are the same object.

We will use the following harmonic averages. For $i, j \in \{1, \ldots, N\}$ define

$$\rho_{ij} = \frac{2\rho_i\rho_j}{\rho_i + \rho_j} \quad \text{and} \quad h_{ij} = \frac{2h_ih_j}{h_i + h_j}. \tag{2.2}$$

We also set $l_{ij} = 2$ when E_{ij} is a common edge (or part of an edge) of $\partial\Omega_i$ and $\partial\Omega_j$. The outward normal derivative on $\partial\Omega_i$ is denoted by $\frac{\partial}{\partial n_i}$.

The discrete problem obtained by the DG method, see [4, 9, 30], is of the form: Find $u_h^* = \{u_{h,i}^*\}_{i=1}^N \in X_h(\Omega)$ such that

$$\widehat{a}_h(u_h^*, v_h) = f(v_h) \quad \text{for all} \quad v_h = \{v_{h,i}\}_{i=1}^N \in X_h(\Omega). \tag{2.3}$$

Here

$$\widehat{a}_h(u, v) = \sum_{i=1}^N \widehat{a}_i(u, v) \text{ and } f(v) = \sum_{i=1}^N \int_{\Omega_i} f v_i dx \tag{2.4}$$

for all $u = \{u_i\}_{i=1}^N, v = \{v_i\}_{i=1}^N \in X_h(\Omega)$. Each local bilinear form $\widehat{a}_i$ is given as a sum of three bilinear forms:

$$\widehat{a}_i(u, v) := a_i(u, v) + s_i(u, v) + p_i(u, v), \tag{2.5}$$

where a_i is the bilinear form,

$$a_i(u, v) := \int_{\Omega_i} \rho_i \nabla u_i \nabla v_i dx, \tag{2.6}$$

the s_i is the symmetric bilinear form,

$$s_i(u, v) := \sum_{E_{ij} \subset \partial\Omega_i} \int_{E_{ij}} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial u_i}{\partial n_i}(v_j - v_i) + \frac{\partial v_i}{\partial n_i}(u_j - u_i) \right) ds$$

and p_i is the penalty bilinear form,

$$p_i(u, v) := \sum_{E_{ij} \subset \partial\Omega_i} \int_{E_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} (u_j - u_i)(v_j - v_i) ds. \tag{2.7}$$

Here ρ_{ij} and l_{ij} are defined in (2.2) and δ is a positive penalty parameter. In order to simplify notation we included the index $j = \partial$ in the definition of the bilinear forms s_i and p_i above. In this case $E_{i\partial} := \partial\Omega_i \cap \partial\Omega$ is an edge of $\partial\Omega_i$ and we set $l_{i\partial} := 1$ and let $v_\partial = 0$ for all $v \in X_h(\Omega)$, and define $\rho_{i\partial} = \rho_i$ and $h_{i\partial} = h_i$. We note that when ρ_{ij} is given by the harmonic average then $\min\{\rho_i, \rho_j\} \leq \rho_{ij} \leq 2\min\{\rho_i, \rho_j\}$.

We also define the positive bilinear forms d_i as

$$d_i(u, v) = a_i(u, v) + p_i(u, v), \tag{2.8}$$

and the broken bilinear form d_h for $X_h(\Omega)$ with weights given by ρ_i and $\frac{\delta}{l_{ij}}\frac{\rho_{ij}}{h_{ij}}$ by

$$d_h(u, v) := \sum_{i=1}^N d_i(u, v). \tag{2.9}$$

For $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ the associated broken norm is then defined by
(2.10)

$$\|u\|_h^2 = d_h(u,u) = \sum_{i=1}^N \left\{ \rho_i \parallel \nabla u_i \parallel_{L^2(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{E_{ij}} (u_i - u_j)^2 ds \right\}.$$

It is known that there exist constants $0 < \delta_0$ and $0 < c < 1$ which do not depend on the h_i, ρ_i and u, such that for $\delta \geq \delta_0$, we have $|s_i(u,u)| < cd_i(u,u)$ and $\sum_i s_i(u,u) < cd_h(u,u)$. We also have the following lemma.

Lemma 2.2. *There exists $\delta_0 > 0$ such that for $\delta \geq \delta_0$ and for all $u \in X_h(\Omega)$ the following inequalities hold:*

$$\gamma_0 d_i(u,u) \leq \widehat{a}_i(u,u) \leq \gamma_1 d_i(u,u),\ i = 1,\ldots,N, \tag{2.11}$$

and

$$\gamma_0 d_h(u,u) \leq \widehat{a}_h(u,u) \leq \gamma_1 d_h(u,u) \tag{2.12}$$

where γ_0 and γ_1 are positive constants independent of the ρ_i, h_i H_i and u.

For the proof of Lemma 2.2 we refer to [9] or [10]. This result implies that the problem (2.3) has a unique solution and it is stable.

Remark 2.3. We note that γ_1/γ_0 in Lemma 2.2 deteriorates when δ gets larger. In practice, however, $\delta \geq \delta_0$ is chosen such that $\delta = O(1)$, therefore, from now on we assume that all the estimates will not depend on δ.

A priori error estimates for the method are optimal for constant coefficient, and also for the case where h_i and h_j are of the same order; see [3, 4, 30].

For discontinuous coefficients and/or when the mesh sizes h_i and h_j are not on the same order, we have the following Lemma 2.4. For the proof, see Theorem 4.2 of [9] and Lemma 2.2 of [10].

Lemma 2.4. *Let u^* and u_h^* be the solutions of (2.1) and (2.3), respectively. For $u^* \in H_0^1(\Omega)$ and $u^*|_{\Omega_i} \in H^{1+\tau}(\Omega_i)$, $i = 1,\ldots,N$, we have*

$$\|u^* - u_h^*\|_h^2 \leq C \sum_{i=1}^N \left(h_i^{1+\tau} \rho_i |u^*|_{H^{1+\tau}(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \frac{h_j}{h_i} h_j^{\tau+1} \rho_j |u^*|_{H^{1+\tau}(\Omega_j)}^2 \right)$$

with $\tau \in (1/2, 1]$ and C is independent of h_i, H_i, ρ_i and u^*.

3. Schur complement problem

In this section we derive the Schur complement bilinear form for the problem (2.3). We first introduce auxiliary notation.
Define $X_i^\circ(\Omega_i)$ as the subspace of $X_i(\Omega_i)$ of functions that vanish on $\partial\Omega_i$. A function $u_i \in X_i(\Omega_i)$ can be represented as

$$u_i = \mathcal{H}_i u_i + \mathcal{P}_i u_i \tag{3.1}$$

where $\mathcal{H}_i u_i \in X_i(\Omega_i)$ is the discrete harmonic part of u_i in the sense of $a_i(.,.)$, see (2.6), i.e., $\mathcal{H}_i u_i$ solves,

$$\begin{cases} a_i(\mathcal{H}_i u_i, v_i) = & 0 \quad \text{for all } v_i \in X_i^\circ(\Omega_i) \\ \mathcal{H}_i u_i = & u_i \quad \text{on} \quad \partial\Omega_i, \end{cases} \tag{3.2}$$

while $\mathcal{P}_i u_i \in X_i^\circ(\Omega_i)$ is the projection of u_i into $X_i^\circ(\Omega_i)$ in the sense of $a_i(.,.)$, i.e.,

$$a_i(\mathcal{P}_i u_i, v_i) = a_i(u_i, v_i) \quad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{3.3}$$

Note that $\mathcal{H}_i u_i$ is the *classical* discrete harmonic part of u_i. Let us denote by $X_h^\circ(\Omega)$ the subspace of $X_h(\Omega)$ defined by $X_h^\circ(\Omega) = X_1^\circ(\Omega_1) \times \cdots \times X_N^\circ(\Omega_N)$ and consider the global projections $\mathcal{H}u := \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\mathcal{P}u := \{\mathcal{P}_i u_i\}_{i=1}^N : X_h(\Omega) \to X_h^\circ(\Omega)$ in the sense of $\sum_{i=1}^N a_i(\cdot,\cdot)$. Hence, a function $u \in X_h(\Omega)$ can then be decomposed as

$$u = \mathcal{H}u + \mathcal{P}u. \tag{3.4}$$

Alternatively to (3.4), a function $u \in X_h(\Omega)$ can be represented as

$$u = \widehat{\mathcal{H}}u + \widehat{\mathcal{P}}u, \tag{3.5}$$

where $\widehat{\mathcal{P}}u = \{\widehat{\mathcal{P}}_i u_i\}_{i=1}^N : X_h(\Omega) \to X_h^\circ(\Omega)$ is the projection in the sense of the original bilinear form $\widehat{a}_h(\cdot,\cdot)$, see (2.4), and $\widehat{\mathcal{H}}u = \{\widehat{\mathcal{H}}_i u\}_{i=1}^N \in X_h(\Omega)$ where $\widehat{\mathcal{H}}_i u$ is the discrete harmonic part of u in the sense of $\widehat{a}_i(.,.)$ defined in (2.5), i.e., $\widehat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$\begin{cases} \widehat{a}_i(\widehat{\mathcal{H}}_i u, v_i) &= 0 \qquad \text{for all } v_i \in X_i^\circ(\Omega_i), \\ \qquad \widehat{\mathcal{H}}_i u &= u_i \qquad \text{on } \partial\Omega_i \\ \qquad \widehat{\mathcal{H}}_i u &= u_j \qquad \text{on every (part of) edge } E_{ji} \subset \partial\Omega_j. \end{cases} \tag{3.6}$$

Here the index j in the last equation of (3.6) runs over all Ω_j and $j = \partial$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j$ and $\overline{\Omega}_i \cap \partial\Omega$ has one-dimensional nonzero measure, respectively. In the latter case, recall that $u_\partial = 0$.

Observe that since $\widehat{\mathcal{P}}_i u_i \in X_i^\circ(\Omega_i)$ we have that for all $v_i \in X_i^\circ(\Omega_i)$,

$$a_i(\widehat{\mathcal{P}}_i u, v_i) = \widehat{a}_h(u, R_i^T v_i),$$

where $R_i^T : X_i^\circ(\Omega_i) \to X_h(\Omega)$ is the standard discrete zero extension operator, i.e., $R_i^T v_i := \{v_j\}_{j=1}^N$, where v_j vanishes for $j \neq i$; see also Section 4 for the definition of other discrete zero extension operators I_i and $\widetilde{I}_i$.

The discrete solution of (2.3) can be decomposed as $u_h^* = \widehat{\mathcal{H}}u_h^* + \widehat{\mathcal{P}}u_h^*$. To compute the projection $\widehat{\mathcal{P}}u_h^*$ we need to solve the following set of standard discrete Dirichlet problems:

$$a_i(\widehat{\mathcal{P}}_i u_h^*, v_i) = f(R_i^T v_i) \qquad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{3.7}$$

Note that these problems, for $i = 1, \dots N$, are local and independent, and so, they can be solved in parallel. This is a precomputational step.

We next formulate the problem for $\widehat{\mathcal{H}}u_h^*$. We first point out that for $v_i \in X_i^\circ(\Omega_i)$ we have

$$\widehat{a}_i(u_i, v_i) = (\rho_i \nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{E_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial v_i}{\partial n}, u_j - u_i\right)_{L^2(E_{ij})}. \tag{3.8}$$

Note that (3.6) has a unique solution. To see this, let us rewrite (3.6) in the form

$$\rho_i (\nabla \widehat{\mathcal{H}}_i u, \nabla \varphi_k^i)_{L^2(\Omega_i)} = -\sum_{E_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial \varphi_k^i}{\partial n}, u_j - u_i\right)_{L^2(E_{ij})} \tag{3.9}$$

where φ_k^i is the nodal basis function of $X_i^\circ(\Omega_i)$ associated with any interior nodal point x_k of the h_i-triangulation of Ω_i. The normal derivative $\frac{\partial \varphi_k^i}{\partial n}$ does not vanish on $\partial\Omega_i$ when x_k is a node of $\tau \in \mathcal{T}_{h_i}(\Omega_i)$ such that $\overline{\tau} \cap \partial\Omega_i \neq \emptyset$. We see that $\widehat{\mathcal{H}}_i u$ is a special extension into Ω_i where u is given on $\partial\Omega_i$ and on all (part of) edges E_{ji} belonging to $\partial\Omega_j$ and common to $\partial\Omega_i$. Therefore, $\widehat{\mathcal{H}}_i u$ depends not only on the values of u_i on $\partial\Omega_i$ but also on the values of u_j given on $E_{ji} = \partial\Omega_i \cap \partial\Omega_j$ and on $E_{\partial i}$ (we already have assumed that $u_\partial = 0$). Note that $\widehat{\mathcal{H}}_i u$ is discrete harmonic except at nodal points close to $\partial\Omega_i$. We will sometimes call $\widehat{\mathcal{H}}_i u$ as the discrete harmonic in a special sense, i.e., in the sense of $\widehat{a}_i(\cdot,\cdot)$.

For $u \in X_h(\Omega)$ observe that (3.6) is obtained from

$$\widehat{a}_h(\widehat{\mathcal{H}}u, v) = 0 \tag{3.10}$$

by taking $v = \{v_i\}_{i=1}^N \in X_h^\circ(\Omega)$. It is easy to see that $\widehat{\mathcal{H}}u = \{\widehat{\mathcal{H}}_i u\}_{i=1}^N$ and $\widehat{\mathcal{P}}u = \{\widehat{\mathcal{P}}_i u_i\}_{i=1}^N$ are orthogonal in the sense of $\widehat{a}_h(.,.)$, i.e.,

$$\widehat{a}_h(\widehat{\mathcal{H}}u, \widehat{\mathcal{P}}v) = 0, \qquad u, v \in X^h(\Omega). \tag{3.11}$$

In addition,

$$\mathcal{H}\widehat{\mathcal{H}}u = \mathcal{H}u \;\text{ and }\; \widehat{\mathcal{H}}\mathcal{H}u = \widehat{\mathcal{H}}u \tag{3.12}$$

since neither $\widehat{\mathcal{H}}u$ nor $\mathcal{H}u$ changes the values of u at the nodes on the boundaries of the substructures Ω_i; see (3.2) and (3.6).

Define

$$\Gamma_h = (\cup_i \partial\Omega_{ih_i}), \tag{3.13}$$

where $\partial\Omega_{ih_i}$ is the set of nodal points of $\partial\Omega_i$. *We note that the definition of* Γ_h *includes the nodes on both sides of every edge or part of edge* $\partial\Omega_i \cap \partial\Omega_j$.

We are now in a position to derive the Schur complement problem for (2.3). Applying the decomposition (3.5) to (2.3) we obtain

$$\widehat{a}_h(\widehat{\mathcal{H}}u_h^* + \widehat{\mathcal{P}}u_h^*, \widehat{\mathcal{H}}v_h + \widehat{\mathcal{P}}v_h) = f(\widehat{\mathcal{H}}v_h + \widehat{\mathcal{P}}v_h)$$

or

$$\widehat{a}_h(\widehat{\mathcal{H}}u_h^*, \widehat{\mathcal{H}}v_h) + 2\widehat{a}_h(\widehat{\mathcal{H}}u_h^*, \widehat{\mathcal{P}}v_h) + \widehat{a}_h(\widehat{\mathcal{P}}u_h^*, \widehat{\mathcal{P}}v_h) = f(\widehat{\mathcal{H}}v_h) + f(\widehat{\mathcal{P}}v_h).$$

Using (3.7) and (3.10) we have

$$\widehat{a}_h(\widehat{\mathcal{H}}u_h^*, \widehat{\mathcal{H}}v_h) = f(\widehat{\mathcal{H}}v_h) \qquad \text{for all } v_h \in X_h(\Omega). \tag{3.14}$$

This is the Schur complement problem for (2.3). We denote by V the space of discrete harmonic functions in the sense of the $\widehat{\mathcal{H}}$, i.e.,

$$V := \left\{ v_h \in X_h(\Omega) :\; v_h \equiv \widehat{\mathcal{H}}v_h \text{ in } \Omega \right\}. \tag{3.15}$$

An important observations is that functions of V are completely determined by its nodal values on Γ_h. That is, in order to define a function in V we need only to specify its values at the nodes of Γ_h.

We rewrite the Schur complement problem as follows: Find $u_h^* \in V$ such that

$$\mathcal{S}(u_h^*, v_h) = g(v_h) \qquad \text{for all } v_h \in V \tag{3.16}$$

where, here and below, $u_h^* \equiv \widehat{\mathcal{H}}u_h^*$ and for u_h and $v_h \in X_h(\Omega)$,

$$\mathcal{S}(u_h, v_h) := \widehat{a}_h(\widehat{\mathcal{H}}u_h, \widehat{\mathcal{H}}v_h) \;\text{ and }\; g(v_h) := f(\widehat{\mathcal{H}}v_h). \tag{3.17}$$

The Schur complement problem (3.16) has a unique solution.

We now introduce some notation and facts to be used later. Let $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and consider $d_i(\cdot,\cdot)$ and $d_h(\cdot,\cdot)$, the bilinear forms defined in (2.8) and (2.9), respectively. First note that for $u \in X_h(\Omega)$, Lemma 2.2 states that

$$\gamma_0 d_h(u,u) \le \widehat{a}_h(u,u) \le \gamma_1 d_h(u,u), \tag{3.18}$$

where γ_0 and γ_1 are positive constants independent of h_i, H_i, ρ_i and u. Additionally, the following lemma shows the equivalence of local energies of discrete harmonic functions in the sense of $\mathcal{H}$ and in the sense of $\widehat{\mathcal{H}}$. The proof can be found in [10].

Lemma 3.1. *For $u \in X_h(\Omega)$ we have*

$$d_i(\mathcal{H}_i u, \mathcal{H}_i u) \le d_i(\widehat{\mathcal{H}}_i u, \widehat{\mathcal{H}}_i u) \le C d_i(\mathcal{H}_i u, \mathcal{H}_i u),\ \ i = 1,\dots,N, \tag{3.19}$$

and

$$d_h(\mathcal{H}u, \mathcal{H}u) \le d_h(\widehat{\mathcal{H}}u, \widehat{\mathcal{H}}u) \le C d_h(\mathcal{H}u, \mathcal{H}u) \tag{3.20}$$

where $\mathcal{H}u = \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\widehat{\mathcal{H}}u = \{\widehat{\mathcal{H}}_i u\}_{i=1}^N$ are defined by (3.2) and (3.6) respectively, and C is a positive constant independent of h_i, u, ρ_i and H_i.

From (3.18) and (3.20) we have

$$\gamma_0 d_h(\mathcal{H}u, \mathcal{H}u) \le \widehat{a}_h(\widehat{\mathcal{H}}u, \widehat{\mathcal{H}}u) \le C\gamma_1 d_h(\mathcal{H}u, \mathcal{H}u) \tag{3.21}$$

and therefore for $\widehat{\mathcal{H}}$, we can take advantages of all the discrete Sobolev norm results known for $\mathcal{H}$ discrete harmonic extensions and for the norm d_h.

4. Notation and the interface condition

In this section we introduce local and global subspaces and extension and restriction operators for functions defined on the interface Γ_h (see (3.13)). Here and below we use the same symbol to denote both piecewise finite element functions and their vector representations. We also introduce a sufficient condition (Assumption 4.1) for designing robust preconditioners and for deriving quasi-optimal bounds for the condition number of the preconditioners. In Section 7 we show numerically that Assumption 4.1 is indeed necessary for robustness.

First we classify substructures according to their position with respect to the boundary $\partial\Omega$. We say that a substructure Ω_i is an *interior substructure* or *floating substructures* if Ω_i does not share an edge with the boundary of Ω. Otherwise, we say it is a *boundary substructure* or *nonfloating substructure*. We denote by $\mathcal{N}_I$ and $\mathcal{N}_B$ the sets of indices of interior and boundary substructures, respectively.

We need to classify the nodes associated to the substructures boundaries according to their relative location with respect to edges and/or part of edges. To this end we introduce special notation which is summarized in Table 1. See Figure 1 for an example. Recall that E_{ij} is a *closed* interval.

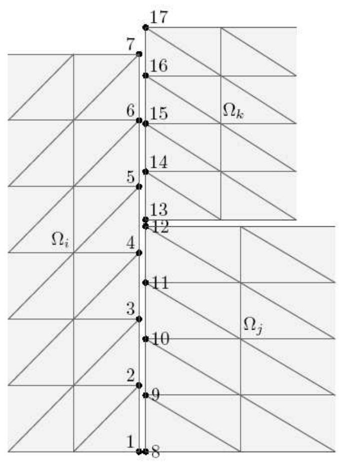
Using the notation in Table 1 we define

$$\Gamma_i = \partial\Omega_{ih_i} \cup \bigcup_{E_{ij}\subset\partial\Omega_i} \overline{E}_{jih_j}. \tag{4.1}$$

The definition of Γ_i is motivated by definition of $\widehat{\mathcal{H}}_i$. Note that Γ_i includes the nodes on Γ_h that are required in the computation of $\widehat{\mathcal{H}}_i$ in (3.6).

TABLE 1. Notation

Notation	Explanation
$\mathring{\Omega}_{ih_i}$	Interior nodes of $\mathcal{T}_{h_i}(\overline{\Omega}_i)$
$\partial\Omega_{ih_i}$	Boundary nodes of $\mathcal{T}_{h_i}(\overline{\Omega}_i)$
E_{ijh_i}	Nodes of $\partial\Omega_{ih_i}$ that are on E_{ij}
∂E_{ijh_i}	Nodes on E_{ijh_i} that are closest to the boundary ∂E_{ij}
$\mathring{E}_{ijh_i}$	Interior nodes of E_{ij} defined by $\mathring{E}_{ijh_i} = E_{ijh_i} \setminus \partial E_{ijh_i}$
$\partial^e E_{ijh_i}$	Extended boundary nodes defined as the union of ∂E_{ijh_i} and the nodal points $y \in \partial\Omega_i \setminus E_{ij}$ closest to $x \in \partial E_{ij}$ when x is not a nodal point
$\overline{E}_{ijh_i}$	Nodes associated with E_{ij} defined by $\overline{E}_{ijh_i} = \mathring{E}_{ijh_i} \cup \partial^e E_{ijh_i}$



$\overline{E}_{ijh_i} = \{1,2,3,4,5\}, \partial E_{ijh_i} = \{1,4\},$
$\mathring{E}_{ijh_i} = \{2,3\}, \partial^e E_{ijh_i} = \{1,4,5\}$

$\overline{E}_{ikh_i} = \{4,5,6,7\}, \partial E_{ikh_i} = \{5,7\},$
$\mathring{E}_{ikh_i} = \{6\}, \partial^e E_{ikh_i} = \{4,5,7\}$

$\overline{E}_{jih_j} = \{8,9,10,11,12\}, \partial E_{jih_j} = \{8,12\},$
$\mathring{E}_{jih_j} = \{9,10,11\}, \partial^e E_{jih_j} = \{8,12\}$

$\overline{E}_{kih_k} = \{13,14,15,16,17\}, \partial E_{kih_k} = \{13,16\},$
$\mathring{E}_{kih_k} = \{14,15\}, \partial^e E_{kih_k} = \{13,16,17\}$

FIGURE 1. An example of node classification on an interface. For the notation see Table 1.

The local subspace W_i is the space of piecewise linear functions, or its vector representation, defined by the nodal values on Γ_i extended via $\widehat{\mathcal{H}}_i$ inside Ω_i, i.e.,

$$W_i := \left\{ v \text{ with nodal values defined on } \mathring{\Omega}_{ih_i} \cup \Gamma_i \text{ and } v \equiv \widehat{\mathcal{H}}_i v \text{ in } \Omega_i \right\}. \tag{4.2}$$

Observe that a function $u^{(i)} \in W_i$ can be represented as

$$u^{(i)} = \{u_l^{(i)}\}_{l \in \#(i)} \quad \text{where} \quad \#(i) = \{i\} \cup \{j : E_{ij} \subset \Omega_i\}.$$

Here $u_i^{(i)}$ and $u_j^{(i)}$ stand for the nodal values of $u^{(i)}$ on $\overline{\Omega}_i$ and on $\overline{E}_{jih_j}$, respectively. According to our conventions, if $i \in \mathcal{N}_B$ and $u^{(i)} \in W_i$ then $u_\partial^{(i)} = 0$ on the fictitious side $E_{\partial i}$.

An important fact is that functions in W_i are completely determined by its nodal values at the nodes in Γ_i defined in (4.1). That is, in order to define a function in W_i we need only to specify its value at the nodes in Γ_i. We will use this fact several times in the rest of the paper.

Now we define the extension operator $\widetilde{I}_i : W_i \to V$. The space W_i is defined in (4.2) and V is defined in (3.15). Given $u^{(i)} \in W_i$, let $\widetilde{I}_i u^{(i)} \in V$ be the only function in V with the following nodal values on Γ_h,

$$\widetilde{I}_i u^{(i)}(x) = \begin{cases} u^{(i)}(x) & \text{if } x \in \Gamma_i \\ 0 & \text{if } x \in \Gamma_h \backslash \Gamma_i, \end{cases} \tag{4.3}$$

and discrete harmonic in the sense of $\widehat{\mathcal{H}}$ inside every substructure Ω_j, $j = 1, \dots, N$. This means that $v = \widetilde{I}_i u^{(i)}$ defines uniquely $v = \{v_j\}_{j=1}^N \in V$ such that $v_j = \widetilde{I}_i u^{(i)}$ on $\partial\Omega_{jh_j}$ for $j = 1, \dots, N$.

Recall that we identify E_{ij} and E_{ji} as being different sides of $\partial\Omega_j \cap \partial\Omega_j$. The mesh on E_{ij} is inherited from $\mathcal{T}_{h_i}(\Omega_i)$ while the mesh on E_{ji} corresponds to $\mathcal{T}_{h_j}(\Omega_j)$; see Remark 2.1 and Table 1. We remind that $\partial\Omega_i \cap \partial\Omega_j$ does not need to be a full edge of Ω_i nor Ω_j.

From each pair $\{E_{ij}, E_{ji}\}$ we label one side as the *master side* (or master) and the other one as the *slave side* (or slave). If E_{ij} is chosen to be the slave then E_{ji} must be the master. If E_{ij} is a slave side we will use the notation δ_{ij} (instead of E_{ij}) to emphasis this fact, while if E_{ij} is a master side we will use the notation γ_{ij}. Note that since we are working with a geometrically nonconforming decomposition of Ω, a part of an edge can be labeled as master while other part of the same edge can be marked as slave. We will use the notation $\gamma_{ijh_i} := E_{ijh_i}$, $\overset{\circ}{\gamma}_{ijh_i} := \overset{\circ}{E}_{ijh_i}$, $\overline{\gamma}_{ijh_i} := \overline{E}_{ijh_i}$ $\partial\gamma_{ijh_i} := \partial E_{ijh_i}$, $\partial^e\gamma_{ijh_i} := \partial^e E_{ijh_i}$ when E_{ij} is a master side. Analogous notation will be used also for a slave side δ_{ij}.

The choice of slave-master sides is such that the *interface condition*, stated next in Assumption 4.1, can be satisfied. Under this assumption, Theorems 5.2, 5.7 and 6.1 below hold with constants C independent of the ρ_i, h_i, H_i and h_i/h_j. This assumption says basically that the coarser meshes h_i should be chosen where the coefficient is larger, and additionally, the master side should be chosen on the side where the coefficient is larger. In terms of accuracy, this condition is satisfied in practice since the solution u^* in general varies less where the coefficient is larger. We note that this condition is similar to the ones adopted in mortar studies for geometrical nonconforming cases; see [22].

Assumption 4.1 (The interface condition)**.** *We say that the values of the coefficient $\{\rho_i\}_{i=1}^N$ and the local mesh sizes $\{h_i\}_{i=1}^N$ satisfy the interface condition if there exist constants β_1 and β_2, of the order $O(1)$, such that for any (part of) edge E_{ij}, one of the following inequalities hold:*

$$\begin{cases} h_i \le \beta_1 h_j \text{ and } \rho_i \le \beta_2 \rho_j & \text{if } E_{ij} \text{ is a slave side, or} \\ h_j \le \beta_1 h_i \text{ and } \rho_j \le \beta_2 \rho_i & \text{if } E_{ij} \text{ is a master side.} \end{cases} \tag{4.4}$$

We say that the Assumption 4.1 holds if β_1 and β_2 are not large.

We associate to each i a weighting diagonal matrix $D^{(i)} = \{D_l^{(i)}\}_{l\in\#(i)}$ on $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$, $i = 1, \cdots, N$. Let x be a node of $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$. Then, the diagonal element of $D^{(i)}$ associated to x is defined by:

- On $\overset{\circ}{\Omega}_{ih_i} \cup \partial\Omega_{i,h_i}$ $(l = i)$

$$D_i^{(i)}(x) = \begin{cases} 0 & \text{if } x \in \overset{\circ}{E}_{ijh_i} \text{ and } E_{ij} \text{ is a slave side} \\ 1 & \text{otherwise,} \end{cases} \tag{4.5}$$

- On $\overline{E}_{jih_j}$ $(l = j)$

$$D_j^{(i)}(x) = \begin{cases} 0 & \text{if } x \in \partial^e E_{jih_j}, \\ 1 & \text{if } x \in \overset{\circ}{E}_{jih_j} \text{ and } E_{ij} \text{ is a master side} \\ 0 & \text{if } x \in \overset{\circ}{E}_{jih_j} \text{ and } E_{ij} \text{ is a slave side,} \end{cases} \tag{4.6}$$

- On $\overline{E}_{i\partial h_i}$

$$D_i^{(i)}(x) = 1 \text{ for all } x \in \overline{E}_{i\partial h_i}.$$

The prolongation operators $I_i : W_i \to V$, $i = 1, \ldots, N$, are defined as

$$I_i = \widetilde{I}_i D^{(i)}. \tag{4.7}$$

Note that the weights used in the definition of the $D^{(i)}$ form a partition of unity on Γ_h, see (3.13), and it follows that

$$\sum_{i=1}^{N} I_i \widetilde{I}_i^T u = u, \tag{4.8}$$

where the $\widetilde{I}_i^T$ stands for the restriction of V to W_i.

Remark 4.2. We can define any value for $D^{(i)}$ on $\overset{\circ}{\Omega}_{ih_i}$ since, as we will see below, the operator of interest is $I_i := \widetilde{I}_i D^{(i)}$ and $\widetilde{I}_i u^{(i)}$ does not depend on the values of $u^{(i)}$ on $\overset{\circ}{\Omega}_{ih_i}$; see (4.3).

Remark 4.3. There are another two alternative ways of defining the diagonal matrices $D^{(i)}$ on Γ_i and still ensuring Theorems 5.2, 5.7 and 6.1 below to hold: 1) On (part of) edges E_{ij}, the values of (4.5) and (4.6) at nodal points x of $\overset{\circ}{E}_{jih_j}$ can be replaced by $\rho_i^\beta/(\rho_i^\beta + \rho_j^\beta)$, $\beta \geq 1/2$ (see reference [32]). To see it note that the $D^{(i)}$ defined in this way also form a partition of unity and (4.8) is satisfied. Unfortunately the estimates given in the Theorems 5.2, 5.7 and 6.1 are valid with constants that depend on the ratios h_i/h_j, so this variant can be used only when h_i and h_j are of the same order. 2) Similarly, on (part of) edges E_{ij}, we can replace (4.5) and (4.6) at nodal points x of $\overset{\circ}{E}_{ijh_i}$ and $\overset{\circ}{E}_{jih_j}$ by $h_i/(h_i + h_j)$. In this case the constants in Theorems 5.2, 5.7 and 6.1 depend on ρ_i/ρ_j, so it can be used only when ρ_i and ρ_j are of the same order, see also below Remark 5.5.

5. Additive preconditioners

To design and analyze additive N-N type methods for solving (3.16) we use the general framework of ASM; see Lemma 5.1 below and [33]. In the Section 5.1 we consider an additive Schwarz operator based on the coarse space $V_{0,I}$, i.e., a coarse space with one degree of freedom per interior substructure and no degrees of freedom for any boundary substructure; see (5.5). Then we consider several variants of this method.

5.1. **Additive Schwarz method with the $V_{0,I}$ coarse space.** We now introduce the local and coarse problems to define the additive Schwarz method $T_{as,I}$.

5.1.1. *Local problems.* Recall the definition of Γ_i in (4.1), the space W_i in (4.2) and that $\mathcal{N}_B$ and $\mathcal{N}_I$ are the sets of indices of interior and boundary substructures, respectively. Define

$$\begin{cases} V_i = V_i(\Gamma_i) = \left\{ u^{(i)} \in W_i : \int_{\partial\Omega_i} u_i^{(i)} = 0 \right\}, & \text{if } i \in \mathcal{N}_I, \\ \\ V_i = V_i(\Gamma_i) = W_i, & \text{if } i \in \mathcal{N}_B, \end{cases} \tag{5.1}$$

i.e., for interior substructures $\{\Omega_i\}_{i\in\mathcal{N}_I}$, V_i is the subspace of W_i consisting of functions with zero average value on $\partial\Omega_i$, while for boundary substructures $\{\Omega_i\}_{i\in\mathcal{N}_B}$, V_i is the whole space W_i. Recall that if $v^{(i)} \in W_i$ (or V_i) then $v^{(i)} \equiv \widehat{\mathcal{H}}_i v^{(i)}$ and if $v \in V$ then $v \equiv \widehat{\mathcal{H}} v$.

For $u^{(i)}, v^{(i)} \in V_i$, $i = 1, \ldots, N$, we define the local bilinear form b_i as

$$b_i(u^{(i)}, v^{(i)}) = \widehat{a}_i(u^{(i)}, v^{(i)}), \tag{5.2}$$

where the bilinear form $\widehat{a}_i$ is defined in (2.5). We define the operators $T_i : V \to V$, $i = 1, \ldots, N$, by defining $\widetilde{T}_i : V \to V_i$ as

$$b_i(\widetilde{T}_i u, v^{(i)}) = \widehat{a}_h(u, I_i v^{(i)}) \text{ for all } v^{(i)} \in V_i, \tag{5.3}$$

and then set $T_i = I_i \widetilde{T}_i$. In matrix form $T_i u$ is given by

$$T_i u = \widetilde{I}_i D^{(i)} (S^{(i)})^{-1} D^{(i)} \widetilde{I}_i^T S u,$$

where the matrices S and $S^{(i)}$ are defined by

$$(v^{(i)})^T S^{(i)} u^{(i)} = \widehat{a}_i(\widehat{\mathcal{H}}_i u^{(i)}, \widehat{\mathcal{H}}_i v^{(i)}), \ \forall u^{(i)}, v^{(i)} \in W_i$$

and

$$v^T S u = \widehat{a}_h(\widehat{\mathcal{H}} u, \widehat{\mathcal{H}} v), \ \forall u, v \in V.$$

It is easy to see, from Lemma (2.2), that these problems are well posed.

5.1.2. *Coarse problem.* Let $e^{(i)} \in W_i$ be the vector with value one at the nodes of Γ_i and on $\overset{\circ}{\Omega}_{ih_i}$. Recall that the prolongation operators $\widetilde{I}_i$ and I_i are defined in (4.3) and (4.7), respectively. Define $\theta_i \in V$, for $i = 1, \ldots, N$, as $\theta_i = \widetilde{I}_i \Theta^{(i)}$ where $\Theta^{(i)} = D^{(i)} e^{(i)}$, hence, $\theta_i = I_i e^{(i)}$. From (4.5) and (4.6) we conclude that

$$\sum_{i=1}^{N} \theta_i = 1 \text{ on } \Gamma_h. \tag{5.4}$$

We consider the following coarse space:

$$V_{0,I} := \operatorname{Span} \{\theta_i\}_{i\in\mathcal{N}_I} \subset V. \tag{5.5}$$

The coarse bilinear form is defined according to

$$b_0(u, v) = \left(1 + \log \frac{H}{h}\right)^{-2} \widehat{a}_h(u, v), \quad u, v \in V_{0,I}. \tag{5.6}$$

We define the projection-like operator $T_0 : V \to V_{0,I}$ as

$$b_0(T_0 u, v^{(0)}) = \widehat{a}_h(u, v^{(0)}) \text{ for all } v^{(0)} \in V_{0,I}. \tag{5.7}$$

Let us denote below $V_0 = V_{0,I}$ and I_0 by the identity operator defined on functions $V_0 \subset V$.

The additive preconditioner is defined by

$$T_{as,I} = \sum_{i=0}^{N} T_i. \tag{5.8}$$

5.1.3. *Condition number estimate for $T_{as,I}$ with the $V_{0,I}$ coarse space.* In this section we state and prove the main result concerning the preconditioner $T_{as,I}$ defined in (5.8) with $V_0 = V_{0,I}$. Note that $T_{as,I}$ is symmetric with respect to the inner product $\widehat{a}_h(\cdot,\cdot)$. From the abstract theory of ASM we have the following abstract lemma.

Lemma 5.1 (See Theorem 2.7 in [33])**.** *Suppose that the following three assumptions hold:*
Assumption i) There exists a constant C_0 such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^{N} I_i u^{(i)}$ with $u^{(i)} \in V_i$, $i = 0, 1, \ldots, N$, such that

$$b_0(u^{(0)}, u^{(0)}) + \sum_{i=1}^{N} b_i(u^{(i)}, u^{(i)}) \le C_0^2 \widehat{a}_h(u, u). \tag{5.9}$$

Assumption ii) There exist constants ϵ_{ij}, $i, j = 1, \ldots, N$, such that for all $u^{(i)} \in V_i$, $u^{(j)} \in V_j$ we have

$$\widehat{a}_h(I_i u^{(i)}, I_j u^{(j)}) \le \epsilon_{ij} \widehat{a}_h(I_i u^{(i)}, I_i u^{(i)})^{1/2} \widehat{a}_h(I_j u^{(j)}, I_j u^{(j)})^{1/2}.$$

Assumption iii) There exists a constant ω such that

$$\widehat{a}_h(I_i u^{(i)}, I_i u^{(i)}) \le \omega b_i(u^{(i)}, u^{(i)}) \ \textit{for all } u^{(i)} \in V_i,\ i = 0, 1, \ldots, N.$$

Then, $T_{as,I}$ is invertible and

$$C_0^{-2} \widehat{a}_h(u, u) \le \widehat{a}_h(T_{as,I} u, u) \le (\rho(\epsilon) + 1)\omega \widehat{a}_h(u, u) \ \textit{for all } u \in V.$$

Here, $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

The following theorem gives the condition number bound for the preconditioner $T_{as,I}$ defined in (5.8) with $V_0 = V_{0,I}$.

Theorem 5.2. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i \in \mathcal{N}_B$, the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then there exist positive constants C_1 and C_2 independent of h_i, H_i, h_i/h_j and the jumps of ρ_i such that*

$$C_1 \widehat{a}_h(u, u) \le \widehat{a}_h(T_{as,I} u, u) \le C_2 \left(1 + \log \frac{H}{h}\right)^2 \widehat{a}_h(u, u) \ \textit{ for all } u \in V. \tag{5.10}$$

Here $\log(H/h) := \max_i \ \log(H_i/h_i)$.

Proof. By the general theory of ASMs we need to check the three key assumptions of Lemma 5.1.

Assumption i) In order to verify (5.9) it is enough to prove (see Lemma 2.2) that for every $u = \{u_i\}_{i=1}^{N} \in V$, there exist $u^{(i)} \in V_i$, $i = 0, \ldots, N$, such that $u = u^{(0)} + \sum_{i=1}^{N} I_i u^{(i)}$ and

$$b_0(u^{(0)}, u^{(0)}) + \sum_{i=1}^{N} b_i(u^{(i)}, u^{(i)}) \le C_0^2 d_h(u, u) \tag{5.11}$$

where C_0 does not depend on h_i, H_i, h_i/h_j and ρ_i.

Recall that $\theta_i = I_i e^{(i)}$ where $e^{(i)}$ has value one at the nodes of Γ_i and $\overset{\circ}{\Omega}_{ih_i}$. See also (4.5), (4.6) and (4.7). Let $u = \{u_i\}_{i=1}^N \in V$ and define

$$u^{(0)} = \sum_{i\in\mathcal{N}_I} \overline{u}_i\theta_i = \sum_{i\in\mathcal{N}_I} I_i\overline{u}_i e^{(i)}, \tag{5.12}$$

with

$$\overline{u}_i = \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} u_i dx,\ i = 1,\dots N. \tag{5.13}$$

Since the operators I_i defined in (4.7) form a partition of unity on Γ_h, see (4.8), we can write

$$u - u^{(0)} = \sum_{i\in\mathcal{N}_I}^{N} I_i(\widetilde{I}_i^T u - \overline{u}_i e^{(i)}) + \sum_{i\in\mathcal{N}_B} I_i(\widetilde{I}_i^T u) = \sum_{i=1}^{N} I_i u^{(i)}, \tag{5.14}$$

where $u^{(i)} := \widetilde{I}_i^T u - \overline{u}_i e^{(i)}$ if $i \in \mathcal{N}_I$, and $u^{(i)} := \widetilde{I}_i^T u$ if $i \in \mathcal{N}_B$. Note that $u^{(i)} \in V_i, i = 1,\cdots,N$.

Note that $u^{(i)}$ can be represented as $u^{(i)} = \{u_l^{(i)}\}_{l\in\#(i)} \in V_i$, for $i = 1,\cdots,N$. For $i \in \mathcal{N}_I$ we have

$$\begin{cases} u_i^{(i)} = u_i - \overline{u}_i e_i^{(i)} = u_i - \overline{u}_i & \text{on } \overline{\Omega}_i, \\ u_j^{(i)} = u_j - \overline{u}_i e_j^{(i)} = u_j - \overline{u}_i & \text{on } E_{ji},\ \text{for all } E_{ij}\subset\partial\Omega_i, \end{cases} \tag{5.15}$$

while for $i \in \mathcal{N}_B$ we have

$$\begin{cases} u_i^{(i)} = u_i & \text{on } \partial\Omega_i, \\ u_j^{(i)} = u_j & \text{on } E_{ji},\ \text{for all } E_{ij}\subset\partial\Omega_i. \end{cases} \tag{5.16}$$

Using Lemma 2.2 we have that for $i = 1,\cdots,N$,

$$\begin{aligned} b_i(u^{(i)},u^{(i)}) &\preceq \rho_i \parallel \nabla\mathcal{H}_i u_i^{(i)} \parallel_{L^2(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \delta\frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(i)} - u_j^{(i)} \parallel_{L^2(E_{ij})}^2 \\ &= \rho_i \parallel \nabla\mathcal{H}_i u_i \parallel_{L^2(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \delta\frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel_{L^2(E_{ij})}^2 . \end{aligned} \tag{5.17}$$

It remains to estimate $b_0(u^{(0)},u^{(0)})$. In Lemma 5.3, see below, we will prove that

$$d_h(u^{(0)},u^{(0)}) \le C\left(1 + \log\frac{H}{h}\right)^2 d_h(u,u), \tag{5.18}$$

and therefore, together with Lemma 2.2 and the definition of b_0 in (5.6), we have that

$$b_0(u^{(0)},u^{(0)}) \le C\widehat{a}_h(u,u) \tag{5.19}$$

where C does not depend on h_i, H_i, h_i/h_j and ρ_i.

Assumption ii) We need to prove that

$$\widehat{a}_h(I_i u^{(i)}, I_j u^{(j)}) \le \varepsilon_{ij} a_h^{1/2}(I_i u^{(i)}, I_i u^{(i)})\ a_h^{1/2}(I_j u^{(j)}, I_j u^{(j)}) \tag{5.20}$$

for $u^{(i)} \in V_i$ and $u^{(j)} \in V_j,\ i,j = 1,\cdots,N$, and that the spectral radius $\varrho(\varepsilon)$ of $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^N$ is bounded. In our case $\varrho(\varepsilon) \leq C$ with constant independent of h_i, H_i, h_i/h_j and ρ_i, $i = 1,\ldots,N$. This follows from the fact that ε_{ij} vanishes when Γ_i and Γ_j do not touch each other.

Assumption iii). We need to prove that for $i = 0,1,\cdots,N$,

$$\widehat{a}_h(I_i u^{(i)}, I_i u^{(i)}) \leq \omega b_i(u^{(i)}, u^{(i)}) \quad \text{for all } u^{(i)} \in V_i \tag{5.21}$$

with $\omega \leq C\,(1+\log(H/h))^2$ where C is a positive constant independent of h_i, H_i, h_i/h_j and the jumps of ρ_i. The proof of (5.21) for $i = 0$ with $\omega = C\,(1+\log(H/h))^2$ follows from the definition of $b_0(\cdot,\cdot)$, while for $i = 1,\ldots,N$, the proof will be presented separately in Lemma 5.4 below. □

The next two auxiliary Lemmas are proven in Appendix A and Appendix B.

Lemma 5.3. *Let the Assumption 4.1 be satisfied. Then for any $u \in V$ and $u^{(0)}$ defined by (5.12), the following inequality holds*

$$d_h(u^{(0)}, u^{(0)}) \leq C\left(1+\log\frac{H}{h}\right)^2 d_h(u,u) \tag{5.22}$$

where the constant C does not depend on h_i, H_i, h_i/h_j and the jumps of ρ_i.

Lemma 5.4. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i \in \mathcal{N}_B$ the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then for $u^{(i)} \in V_i$, $i = 1,\ldots,N$, we have*

$$\widehat{a}_h(I_i u^{(i)}, I_i u^{(i)}) \leq C\left(1+\log\frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}), \tag{5.23}$$

where C does not depend on h_i, H_i, h_i/h_j and the jumps of ρ_i.

Remark 5.5 (Weak interface condition for general distribution of coefficients). We note that there exist distributions of coefficients $\{\rho_i\}_{i=1}^N$ and mesh sizes $\{h_i\}_{i=1}^N$ where Assumption 4.1 does not hold for any choice of slave-master sides, i.e., slave and master sides can not be chosen properly so that neither β_1 nor β_2 is bounded. We note, however, that always can choose slave and master sides such that the following weaker Assumption 5.6 (stated next) holds, see also Remark 4.3.

Assumption 5.6. *We say that the values of the coefficient $\{\rho_i\}_{i=1}^N$ satisfy the weak interface condition if there exists a constant β_2 of the order $O(1)$, such that for any (part of) edge E_{ij}, one of the following inequalities hold:*

$$\begin{cases} \rho_i \leq \beta_2\rho_j & \text{if } E_{ij} \text{ is a slave side, or} \\ \rho_j \leq \beta_2\rho_i & \text{if } E_{ij} \text{ is a master side.} \end{cases} \tag{5.24}$$

Note that Assumption 5.6 can always be satisfied with $\beta_2 = 1$ if we place the mortar side on the side where the coefficient is larger. Note that Assumption 4.1 implies Assumption 5.6.

Under the Assumption 5.6 condition, it is easy to show that the constants C in Lemma 5.3 and also in Lemma 5.4, do not depend on h_i, H_i, and the jumps of ρ_i, however, it depend on the maximum value of h_i/h_j among all (part of) edges E_{ij} in Ω. This implies that the lower bound estimation for C_1 and C_2 for Theorem 5.2 does not depend on h_i, H_i, and the jumps of ρ_i, but it might deteriorate when β_1 in Assumption 4.1 gets larger. Similar results also extend to all the preconditioners that are introduced later in this paper for the case that the Assumption 5.6 condition is satisfied.

5.2. **Additive Schwarz method with the $V_{0,I\cup B}$ coarse space.** We recall that the upper bound $C(1+\log H/h)^2$ in Theorem 5.2 requires that $|\partial\Omega_i \cap \partial\Omega| \asymp H_i$ for all $i \in \mathcal{N}_B$. Without this condition we obtain an upper bound $C(1+\log H/h)^3$; see Remark B.1. To obtain an upper bound $C(1+\log H/h)^2$ without this condition, we enhance the coarse space $V_{0,I}$, see (5.5), by adding boundary coarse basis functions, i.e., we use the richer coarse space

$$V_{0,I\cup B} := \text{Span}\,\{\theta_i\}_{i\in(\mathcal{N}_I\cup\mathcal{N}_B)}\,. \tag{5.25}$$

The additive preconditioner is then defined by

$$T_{as,I\cup B} = \sum_{i=0}^{N} T_i, \tag{5.26}$$

where the T_0 is defined as in (5.7) except that now we replace $V_{0,I}$ by $V_{0,I\cup B}$. We obtain the following theorem.

Theorem 5.7. *Let the Assumption 4.1 be satisfied. Then there exist positive constants C_1 and C_2 independent of h_i, H_i, h_i/h_j and the jumps of ρ_i such that*

$$C_1\widehat{a}_h(u,u) \leq \widehat{a}_h(T_{as,I\cup B}u,u) \leq C_2\left(1+\log\frac{H}{h}\right)^2 \widehat{a}_h(u,u) \quad \textit{for all } u \in V. \tag{5.27}$$

Here $\log(H/h) := \max_i\ \log(H_i/h_i)$.

Proof. Use that $V_{0,I} \subset V_{0,I\cup B} \subset V$ and repeat the proof of Theorem 5.2 with the discussion in Remark B.1. □

Remark 5.8. If size of any E_{ij} and $E_{i\partial}$ is of the order of H_i, we can use the following coarse bilinear form,

$$\widetilde{b}_0(u,v) = \left(1+\log\frac{H}{h}\right)^{-1} \widehat{a}_h(u,v), \quad u,v \in V_0 \tag{5.28}$$

and still keep the two logarithmic factors in the final results (5.10) and (5.27) of Theorem 5.2 and Theorem 5.7, respectively. In such a case, it is easy to see that (5.22) in Lemma 5.3 will hold with only one logarithmic factor; see the discussions in (A.8) and (A.10).

Remark 5.9. We point out that all the bilinear forms $b_i, i = 0,\cdots,N$, considered until now were based on exact solvers, i.e., based on the bilinear forms $\widehat{a}_h(\widehat{\mathcal{H}}u^{(0)},\widehat{\mathcal{H}}u^{(0)})$ and $\widehat{a}_i(\widehat{\mathcal{H}}_iu^{(i)},\widehat{\mathcal{H}}_iu^{(i)})$. We note that, due to Lemma 2.2 and Lemma 3.1, all the results will still hold if we replace those bilinear forms by $d_h(\mathcal{H}u^{(0)},\mathcal{H}u^{(0)})$, and $d_i(\mathcal{H}_iu^{(i)},\mathcal{H}_iu^{(i)})$, respectively.

6. Hybrid preconditioners

In this section we design and analyze a hybrid type (BDD) method for solving (3.16); see [28, 33]. We consider the hybrid version of $T_{as,I}$, see (5.8). The hybrid version of $T_{as,I\cup B}$, see (5.26), can be treated similarly; see [11].

6.1. **The method.** Recall the definition of the Γ_i in (4.1), the spaces W_i in (4.2), the local subspaces V_i in (5.1) and the coarse subspace $V_0 = V_{0,I}$ in (5.5). Consider the bilinear forms $b_i, i = 1, \cdots, N$, defined in (5.2).

Now define bilinear form a_0 as the exact bilinear form $\widehat{a}_h$, i.e.,

$$a_0(u, v) = \widehat{a}_h(u, v), \quad u, v \in V_0, \tag{6.1}$$

with $\widehat{\mathcal{H}}$ defined in (3.6). Introduce the coarse orthogonal projection $P_0 : V \to V_0$ defined by

$$a_0(P_0 u, v) = \widehat{a}_h(u, v) \quad \text{for all } v \in V_0. \tag{6.2}$$

The hybrid method is defined as (see [33])

$$T_{hyb,I} = P_0 + (I - P_0)\left(\sum_{i=1}^{N} T_i\right)(I - P_0), \tag{6.3}$$

where the operators T_i were defined as $T_i = I_i\widetilde{T}_i$ with $\widetilde{T}_i$ defined by (5.3), $i = 1, \ldots, N$.

Let the subspace $V_0^{\perp} \subset V$ consists of functions $w \in V$ such that $\widehat{a}_h(w, v_0) = 0$, for all $v_0 \in V_0$. It is easy to check that if $w \in V_0^{\perp}$ then $T_{hyb,I}w \in V_0^{\perp}$. The PCG algorithm for solving $T_{hyb,I}v = w$, $w \in V_0^{\perp}$, searches for the best approximation to the solution in the Krylov subspace generated by powers of $T_{hyb,I}$ applied to w. Assume that the goal is to solve $Su = g$, where $u = u_h^*$, see (3.16). We replace this equation by $T_{hyb,I}u = \widetilde{g}$ where $\widetilde{g} = T_{hyb,I}u$, and compute $u_0 = P_0 u$. The computations of $\widetilde{g}$ and u_0 can be obtained directly from g without the knowledge of u by (5.3) and (6.2), respectively; see also [33]. Note that in our case $u = v + u_0$ and $w = T_{hyb,I}u - P_0 u$ belongs to $V_0^{\perp}$. Then we can solve $T_{hyb,I}v = w$ using the PCG algorithm operated on the subspace $V_0^{\perp}$.

6.2. **Condition number estimate for $T_{hyb,I}$.** From the analysis of the additive method $T_{as,I}$ developed in Theorem 5.2 we can derive an analysis for the hybrid method $T_{hyb,I}$. Observe that in both methods we have considered the same local and coarse spaces. Note also that in the design of the hybrid method $T_{hyb,I}$ we have considered the bilinear form $a_0(\cdot, \cdot)$ defined in (6.1) rather than the bilinear form $b_0(\cdot, \cdot)$ defined in (5.6). These two bilinear forms differ only from each other by a scaling factor. For both methods we have considered the same local bilinear forms $b_i(\cdot, \cdot)$ defined in (5.2).

Theorem 6.1. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i = 1, \cdots, N$, the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then there exists a positive constant C independent of h_i, H_i, h_j/h_j and the jumps of ρ_i such that*

$$\widehat{a}_h(u, u) \le \widehat{a}_h(T_{hyb,I}u, u) \le C\left(1 + \log\frac{H}{h}\right)^2 \widehat{a}_h(u, u) \text{ for all } u \in V_0^{\perp}. \tag{6.4}$$

Here $\log(H/h) := \max_i \ \log(H_i/h_i)$.

Proof. Upper Bound: Using Rayleigh quotient arguments and properties of the orthogonal projection P_0, i.e., that $(I - P_0)P_0 = 0$, we obtain

$$\lambda_{\max}(T_{hyb,I_{|V_0^{\perp}}}) = \max_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h(T_{hyb,I}u, u)}{\widehat{a}_h(u, u)}$$

$$= \max_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h(\sum_{i=1}^N T_i u, u)}{\widehat{a}_h(u,u)} = \max_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h([\gamma P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)}$$

$$\leq \max_{u \in V \setminus \{0\}} \frac{\widehat{a}_h([\gamma P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)} = \lambda_{\max}(T_{as,I}),$$

where $\gamma = \left(1 + \log \frac{H}{h}\right)^{-2}$ and $T_{as,I}$ defined in (5.8). Hence, the upper bound follows from the upper bound of Theorem 5.2.

Lower Bound: We obtain

$$\lambda_{\min}(T_{hyb,I}{}_{|V_0^{\perp}}) = \min_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h(T_{hyb,I} u, u)}{\widehat{a}_h(u,u)}$$

$$= \min_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h(\sum_{i=1}^N T_i u, u)}{\widehat{a}_h(u,u)} = \sup_{\epsilon > 0} \min_{u \in V_0^{\perp} \setminus \{0\}} \frac{\widehat{a}_h([\epsilon P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)}$$

$$\geq \sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{\widehat{a}_h([\epsilon P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)}.$$

It remains to show

$$\sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{\widehat{a}_h([\epsilon P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)} \geq 1. \tag{6.5}$$

Let $\epsilon > 0$ be fixed. A lower bound estimation for $\epsilon P_0 + \sum_{i=1}^N T_i$ can be obtained from the general theory of ASMs where we need to check the Assumption i) of Lemma 5.1. To check this assumption, let $u \in V$ and consider the same decomposition $\sum_{i=1}^N I_i u^{(i)} = u$ described in the proof of Theorem 5.2, i.e., $u^{(0)}$ defined in (5.12) and the $u^{(i)}$, $i = 1, \cdots, N$, defined in (5.14). Using the same steps of the proof of Theorem 5.2 we obtain

$$\epsilon \widehat{a}_h(u_0, u_0) \leq C\epsilon(1 + \log \frac{H}{h})^2 \widehat{a}_h(u,u),$$

and

$$\sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) = \widehat{a}_h(u,u).$$

Note that to obtain this equality we do not use d_i as in (5.17). Instead, we work with $\widehat{a}_i$ and we get an equality in (5.17) with right-hand side equal to $\widehat{a}_i$. Summing these equalities we get the above estimates. Hence, we obtain

$$\epsilon \widehat{a}_h(u_0, u_0) + \sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) \leq \left(1 + C\epsilon(1 + \log \frac{H}{h})^2\right) \widehat{a}_h(u,u),$$

and therefore

$$\sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{\widehat{a}_h([\epsilon P_0 + \sum_{i=1}^N T_i]u, u)}{\widehat{a}_h(u,u)} \geq \sup_{\epsilon > 0} \left(1 + C\epsilon(1 + \log \frac{H}{h})^2\right)^{-1} = 1.$$

$\square$

7. Numerical experiments

In this section, we present numerical results for the preconditioners introduced in (5.8), (5.26) and (6.3), for the geometrically nonconforming case. Similar results have been obtained for the geometrically conforming case.
We show that the bounds of Theorems 5.2, 5.7 and 6.1 are reflected in the numerical tests. In particular we show that the interface condition (Assumption 4.1) is necessary and sufficient.

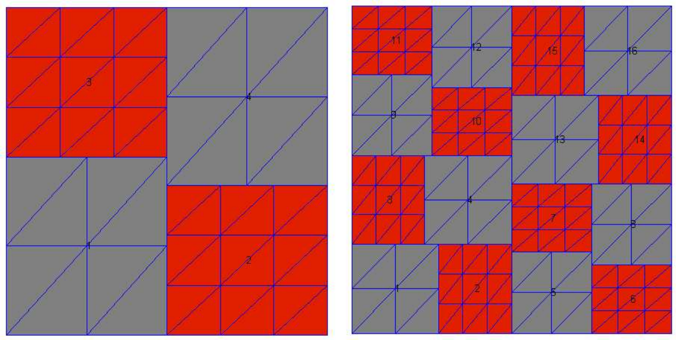


Figure 2. Geometrically nonconforming partition.

We consider the domain $\Omega = (0,1)^2$ and divide it into $N = M \times M$ rectangular geometrically nonconforming substructures Ω_i as in Figure 2. In each substructure, the next level of refinement is obtained from a regular conforming 2×2 rectangular refinement by enlarging (or decreasing) the width or high of some rectangles by a factor $fac = 1 + 1/23$ (or $1 - 1/23$) for $M = 2, 4$; see Figure 2. To obtain substructures with $M = 8$ and $M = 16$ we subdivide each of substructures for the case $M = 4$ (right picture of Figure 2) equally in 2×2 and 4×4 subdomains, respectively. Note that $H \simeq 1/M$.

Inside each substructure Ω_i we generate a structured triangulation with n_i subintervals in each coordinate direction and apply the discretization presented in Section 2 with $\delta = 4$. In the numerical experiments we use a red and black checkerboard type of substructure partition starting with the lower left corner substructure as a black type substructure. On the black substructures we let $n_b = 2 * 2^{L_b}$ and on the red substructures we let $n_r = 3 * 2^{L_r}$, where L_b and L_r are integers denoting the number of refinements inside each substructure Ω_i. Hence, since the size of each substructure is $O(1/M)$ then the mesh sizes are $h_b \simeq \frac{1}{Mn_b}$ and $h_r \simeq \frac{1}{Mn_r}$, respectively. We solve the second order elliptic problem $-\mathrm{div}(\rho(x)\nabla u^*(x)) = 1$ in Ω with homogeneous Dirichlet boundary conditions.

7.1. **Hybrid preconditioner.** In the first test we consider the constant coefficient case $\rho_r = \rho_b = 1$. We consider different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$, therefore, keeping constant the mesh ratio $h_b/h_r \asymp 3/2$. We test two different choices of masters and slaves:

(1) We place the *master on the black substructure side* of the edge or part of edge $\partial\Omega_i \cap \partial\Omega_j$ in the case that Ω_i and Ω_j are two different colors substructures, and place on the most north-east substructure side otherwise.

(2) We place the *master on the red substructure side* of the edge or the part of edge $\partial\Omega_i \cap \partial\Omega_j$ in the case that Ω_i and Ω_j are two different colors substructures, and place on the most south-west substructure side otherwise.

Table 2 lists the number of PCG iterations and in parenthesis the condition number estimate and the smallest eigenvalue of the preconditioned system. We note that for constant coefficients and fixed mesh ratio $h_b \asymp h_r$, the interface condition (Assumption 4.1) is satisfied with $\beta_1 \preceq 3/2$ and $\beta_2 = 1$. As expected by Theorem 6.1, the condition numbers appear to be independent of the number of substructures and grow by a polylogarithmical factor of the number of subdomains when the size of the local problems increases. We recall that Assumption 4.1 implies Assumption 5.6.

TABLE 2. Hybrid preconditioner $T_{hyb,I}$: number of iterations, condition number and smallest eigenvalue (in parenthesis) for different sizes of coarse and local problems and with constant coefficient $\rho_b = \rho_r = 1$ and $L_b = L_r$. . Top: Master/Slave choice (1). Bottom: Master/Slave choice (2)

$L_r \downarrow M \rightarrow$	2	4	8	16
0	12 (6.22,1.00)	18 (7.83,1.03)	20 (9.15,1.01)	22 (12.39,1.01)
1	16 (7.95,1.00)	24 (11.74,1.02)	27 (14.63,1.01)	26 (13.97,1.00)
2	20 (16.03,1.05)	29 (18.87,1.01)	31 (19.64,1.00)	31 (19.43,1.00)
3	22 (20.24,1.06)	32 (24.13,1.00)	35 (23.55,1.00)	34 (23.64,1.00)

$L_r \downarrow M \rightarrow$	2	4	8	16
0	10 (8.21,1.00)	19 (9.06,1.02)	22 (10.73,1.01)	23 (11.39,1.01)
1	14 (10.06,1.00)	25 (13.61,1.01)	27 (14.67,1.01)	27 (14.99,1.00)
2	20 (17.83,1.03)	31 (20.43,1.01)	32 (19.53,1.00)	32 (19.02,1.00)
3	20 (19.55,1.05)	33 (25.12,1.01)	37 (24.68,1.00)	36 (24.21,1.00)

We now consider the discontinuous coefficients case where we set $\rho_b = 1$ on the black substructures and we vary ρ_r on the red substructures. The substructures partition is kept fixed to 4×4. Table 3 lists the results on runs for different values of ρ_r and for different levels of refinements L_r on the red substructures. On the black substructures $n_b = 2$ is kept fixed. We test the same two choices of masters and slaves as in the first experiment. It is easy to see in Table 3 that when the interface condition (Assumption 4.1) holds, i.e. when β_1 and β_2 are not large, then the preconditioner is robust. If the Assumption 4.1 does not hold, however we still choose the master side where the coefficient is larger, i.e., the Assumption 5.6 holds with $\beta_2 = 1$, the performance of the preconditioner is still robust with respect to the coefficients but depends on the mesh ratio h_b/h_r, see Remark 5.5.

7.2. **Additive preconditioner.** For the additive preconditioners we repeat the experiments done for the hybrid preconditioner in the geometrically nonconforming case. As before we consider the constant coefficient case $\rho_r = \rho_r = 1$, the mesh ratio $h_b/h_r \asymp 3/2$. Table 4 shows that the condition numbers appear to be independent of the number of substructures and grow by a polylogarithmical factor when the size of the local problems increases. As expected by Theorem 5.7, Table 5 shows that condition numbers do not change much when we replace $T_{as,I}$ to $T_{as,I\cup B}$.

Now consider the discontinuous coefficients case. On the black substructures we consider $\rho_b = 1$ and on the red substructures we vary ρ_r. The substructure partition

TABLE 3. Hybrid preconditioner $T_{hyb,I}$: number of iterations and condition numbers (in parenthesis) for different values of the coefficient ρ_r and local meshes with L_r refinements) on the red substructures. On black substructures the coefficient $\rho_b = 1$ and $L_b = 0$ are kept fixed. The substructure partition is also kept fixed to 4×4. Top: Master/Salve choice (1). Bottom: Master/Slave choice (2).

$L_r \downarrow \rho_r \rightarrow$	1000	10	1	0.1	0.001
Assum. 4.1	No	No	Yes	Yes	Yes
Assum. 5.6	No	No	Yes	Yes	Yes
0	93(3069.53)	34(34.84)	18(7.83)	18(8.93)	18(9.73)
1	120(4530.84)	43(50.36)	21(10.35)	19(9.60)	19(10.45)
2	175(4990.32)	48(54.73)	23(14.81)	20(15.60)	19(16.24)
3	235(6496.58)	53(69.84)	25(17.54)	20(17.41)	19(18.12)
4	336(7542.38)	57(79.24)	26(20.02)	21(20.05)	19(20.98)

$L_r \downarrow \rho_r \rightarrow$	1000	10	1	0.1	0.001
Assum. 4.1	No	No	No	No	No
Assum. 5.6	Yes	Yes	Yes	No	No
0	17 (11.62)	19 (10.57)	19 (9.06)	27 (27.08)	68 (2392.31)
1	20 (16.87)	21 (15.72)	23 (14.04)	28 (27.32)	70 (2297.89)
2	22 (27.79)	25 (26.21)	27 (23.88)	34 (32.49)	82 (2227.64)
3	27 (49.85)	30 (47.31)	34 (43.06)	41 (52.45)	92 (2194.14)
4	33 (94.07)	40 (89.53)	42 (81.38)	50 (94.33)	109 (2180.58)

TABLE 4. Additive preconditioner $T_{as,I}$: number of iterations, condition numbers and smallest eigenvalues (in parenthesis) for different sizes of coarse and local problems and constant coefficient $\rho_b = \rho_r = 1$ and $L_b = L_r$. Top: Master/Salve choice (1). Bottom: Master/Salve choice (2).

$L_r \downarrow M \rightarrow$	2	4	8	16
0	11 (7.32,1.00)	23 (36.03,0.40)	38 (44.03,0.32)	43 (45.63,0.32)
1	17 (15.01,1.00)	31 (40.09,0.53)	39 (41.87,0.45)	41 (42.06,0.45)
2	22 (20.19,1.03)	35 (47.28,0.63)	41 (49.39,0.56)	44 (48.77,0.56)
3	23 (23.76,1.05)	37 (48.77,0.71)	43 (54.00,0.65)	46 (53.90,0.65)

$L_r \downarrow M \rightarrow$	2	4	8	16
0	10 (10.20,1.00)	25 (40.26,0.38)	44 (60.71,0.29)	49 (62.10,0.28)
1	15 (13.38,1.00)	32 (39.46,0.53)	43 (56.49,0.41)	46 (55.66,0.42)
2	20 (20.84,1.03)	35 (46.94,0.61)	44 (58.28,0.52)	47 (57.38,0.53)
3	22 (21.24,1.05)	38 (50.15,0.70)	46 (62.50,0.62)	49 (61.68,0.62)

is kept fixed to 4×4. Table 6 lists the results on runs for different values of ρ_r and for different levels of refinements L_r on the red substructures. On the black substructures $n_b = 2$ is kept fixed, i.e., $L_b = 0$. It is easy to see in Table 6 that when the interface condition Assumption 4.1 holds the algorithm is robust with respect to the coefficients and mesh ratio h_b/h_r. When the Assumption 4.1 does not hold, however the Assumption 5.6 holds, i.e., when the master side is chosen where the coefficient is larger, as expected from the Remark 5.5, the algorithm is still robust with respect to the coefficients but depends on the mesh ratio h_b/h_r.

TABLE 5. Additive preconditioner $T_{as,I\cup B}$: number of iterations, condition numbers abd smallest eigenvalue (in parenthesis) for different sizes of coarse and local problems and constant coefficient $\rho_b = \rho_r = 1$ and $L_b = L_r$. Top: Master/Slave choice (1). Bottom: Master/Slave choice (2).

$L_r \downarrow M \rightarrow$	2	4	8	16
0	13 (8.20,1.00)	23 (36.99,0.44)	38 (43.54,0.36)	41 (44.95,0.35)
1	18 (17.25,1.00)	31 (44.82,0.55)	38 (43.91,0.48)	40 (43.90,0.47)
2	23 (22.83,1.05)	35 (53.68,0.63)	40 (50.81,0.57)	42 (50.52,0.57)
3	25 (27.10,1.06)	37 (55.76,0.71)	42 (55.32,0.65)	43 (55.36,0.65)

$L_r \downarrow M \rightarrow$	2	4	8	16
0	11 (11.10,1.00)	25 (39.12,0.43)	43 (57.80,0.32)	46 (59.35,0.32)
1	16 (15.21,1.00)	31 (41.90,0.55)	41 (57.60,0.43)	44 (56.93,0.44)
2	23 (23.06,1.03)	35 (49.45,0.62)	43 (59.39,0.53)	45 (58.80,0.53)
3	24 (23.77,1.05)	36 (51.59,0.71)	44 (63.48,0.62)	45 (62.86,0.62)

TABLE 6. Additive preconditioner $T_{as,I}$: number of iterations and condition numbers (in parenthesis) for different values of coefficient ρ_r and the local mesh refinements L_r on the red substructures. The coefficient and the local mesh sizes on the black substructures are kept fixed to $\rho_b = 1$ and $L_b = 1$. The number of substructures are also kept fixed to 4×4. Top: Master/slave choice (1). Bottom: Master/Slave choice (2).

$L_r \downarrow \rho_r \rightarrow$	1000	10	1	0.1	0.001
Assum. 4.1	No	No	Yes	Yes	Yes
Assum. 5.6	No	No	Yes	Yes	Yes
0	129 (225164.57)	47 (205.41)	23 (36.03)	22 (23.75)	21 (24.41)
1	209 (294622.11)	59 (333.17)	26 (42.50)	21 (23.05)	22 (22.86)
2	289 (291958.12)	65 (462.11)	28 (54.06)	23 (36.76)	22 (35.45)
3	395 (289006.68)	71 (621.10)	31 (65.35)	23 (39.06)	21 (36.61)
4	514 (273591.72)	76 (790.20)	32 (78.79)	24 (44.24)	21 (40.29)

$L_r \downarrow \rho_r \rightarrow$	1000	10	1	0.1	0.001
Assum. 4.1	No	No	No	No	No
Assum. 5.6	Yes	Yes	Yes	No	No
0	20 (38.67)	23 (30.12)	25 (40.26)	39 (155.01)	102 (169137.54)
1	25 (91.50)	27 (58.44)	31 (74.73)	41 (182.61)	106 (111142.92)
2	31 (207.09)	36 (128.70)	37 (147.46)	45 (238.01)	121 (67089.50)
3	37 (400.62)	43 (302.86)	50 (365.20)	54 (394.09)	136 (39583.97)
4	45 (808.87)	54 (835.62)	63 (1067.69)	67 (836.05)	153 (24664.06)

8. CONCLUSIONS

In this paper we considered a discontinuous Galerkin discretization for second order elliptic equations with discontinuous coefficients and nonmatching meshes on geometrically nonconforming substructures. We designed and analyzed Neumann-Neumann methods of additive and additive-multiplicative type. We proved that these methods are almost optimal and very well suited for parallel computations. The coarse space is constructed using a special partition of unity. The rate of convergence of both methods are polylogarithmically with respect to the local mesh size, and does not depend on the number of substructures and on the jumps of coefficient. The numerical tests confirm the theoretical results. The methods can be straightforwardly extended to 3-D cases. Despite of the fact the extension of our results to the 3-D case is easy, it needs a lot of details and therefore is not discussed in the paper. It will be considered separately.

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Appendix A. Proof of Lemma 5.3

To avoid the proliferation of constants, we will use sometimes the notation $A \preceq B$ to represent the inequality $A \leq (constant)B$, and $A \asymp B$ if $A \preceq B$ and $B \preceq A$, where the $(constant)$ does not depend on H_i, h_i, h_i/h_j and ρ_i.

Let us denote $u^{(0)} = \{u_i^{(0)}\}_{i=1}^N$. By Lemma 3.1 it is enough to prove the estimate (5.22) for $\mathcal{H}u^{(0)} = \{\mathcal{H}_i u_i^{(0)}\}_{i=1}^N$; see (3.2). Here we denote $\mathcal{H}u^{(0)}$ by $u^{(0)}$ in order to simplify the notation. We have

(A.1)

$$d_h(u^{(0)}, u^{(0)}) = \sum_{i=1}^N \left\{ \rho_i \parallel \nabla u_i^{(0)} \parallel_{L^2(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel_{L^2(E_{ij})}^2 \right\}.$$

Now we will bound each term in the sum above. For simplicity of the presentation we split each summand above in two terms

$$\left\{ \rho_i \parallel \nabla u_i^{(0)} \parallel_{L^2(\Omega_i)}^2 \right\} + \left\{ \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel_{L^2(E_{ij})}^2 \right\} = \{\mathrm{I}\} + \{\mathrm{II}\} \tag{A.2}$$

and bound each one separately. To finish the proof of the lemma we proceed as follows,

1. We bound term I in (A.2) by considering the two cases
   - (a) $i \in \mathcal{N}_I$ (see Section A.1).
   - (b) $i \in \mathcal{N}_B$ (see Section A.2).
2. We bound term II in (A.2) (see Section A.3).

A.1. **Bound of I in (A.2) for interior substructures.**

Let $i \in \mathcal{N}_I$. From the definition of $u^{(0)}$ in (5.12) we see that on $\partial\Omega_i$

$$u_i^{(0)} = \overline{u}_i \Theta_i^{(i)} + \sum_{\delta_{ij}\subset\partial\Omega_i} \overline{u}_j \Theta_i^{(j)} - \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j \Theta_i^{(j)}. \tag{A.3}$$

It is easy to see from (4.6) that when $\delta_{ij} = E_{ij}$ is a slave side and $\overset{\circ}{E}_{ijh_i}$ is empty then $\Theta_i^{(j)}$ vanishes. Hence, we consider only the cases in (A.3) when $\overset{\circ}{E}_{ijh_i}$ is not empty, and hence from the definition of $\overset{\circ}{E}_{ijh_i}$ we have $h_i \preceq |E_{ij}|$, where

$|E_{ij}|$ denotes the size (the one-dimensional Lebesgue measure) of E_{ij}. In general, $|E_{ij}|$ can be very tiny due to the geometrically nonconformity of the Ω_i partition, however, this is not the case when $\mathring{E}_{ijh_i}$ is not empty. Additionally, because E_{ij} is a slave side and the Assumption 4.1 hypothesis holds, we have $h_i \asymp h_{ij} \preceq h_j$. From (5.4) we have

$$u_i^{(0)} - \overline{u}_i = \sum_{\delta_{ij}\subset\partial\Omega_i} (\overline{u}_j - \overline{u}_i)\Theta_i^{(j)} - \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j\Theta_i^{(j)} \quad \text{on } \partial\Omega_i. \tag{A.4}$$

Using (A.4) we obtain

$$\begin{aligned} \| \nabla u_i^{(0)} \|^2_{L^2(\Omega_i)} &= \| \nabla(u_i^{(0)} - \overline{u}_i) \|^2_{L^2(\Omega_i)} \\ &\preceq \left(1 + \log\frac{H_i}{h_i}\right)\left(\sum_{\delta_{ij}\subset\partial\Omega_i} (\overline{u}_i - \overline{u}_j)^2 + \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j^2\right) \end{aligned} \tag{A.5}$$

where we have used the following extension theorem $\| \nabla\Theta_i^{(j)} \|^2_{L^2(\Omega_i)} \preceq \| \Theta_i^{(j)} \|^2_{H^{1/2}_{00}(\delta_{ij})}$ and the discrete inequality (see [33])

$$\| \Theta_i^{(j)} \|^2_{H^{1/2}_{00}(\delta_{ij})} \preceq \left(1 + \log\frac{H_i}{h_i}\right).$$

In order to finish the estimation of term I in (A.2) when $i \in \mathcal{N}_I$ we only need to bound the two terms in the right hand side of (A.5). We do that as follows,

- First term in (A.5). Denote

$$\overline{u}_{ij} = \frac{1}{|E_{ij}|}\int_{E_{ij}} u_i ds \quad \text{and} \quad \overline{u}_{ji} = \frac{1}{|E_{ji}|}\int_{E_{ji}} u_j ds. \tag{A.6}$$

  Note that $h_{ij} \asymp h_i \preceq |E_{ij}|$ and so

$$(\overline{u}_{ij} - \overline{u}_{ji})^2 = \frac{1}{|E_{ij}|^2}(u_i - u_j, 1)^2_{L^2(E_{ij})} \preceq \frac{1}{h_i}\|u_i - u_j\|^2_{L^2(E_{ij})}.$$

  By the discrete and Poincaré inequalities, and using again that $h_i \preceq |E_{ij}|$ we obtain

$$(\overline{u}_i - \overline{u}_{ij})^2 \preceq \left(1 + \log\frac{H_i}{h_i}\right) \| \nabla u_i \|^2_{L^2(\Omega_i)}. \tag{A.7}$$

  Using the above estimates we obtain

$$\begin{aligned} (\overline{u}_i - \overline{u}_j)^2 &\preceq (\overline{u}_i - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_j)^2 \\ &\preceq \left(1 + \log\frac{H_i}{h_i}\right) \| \nabla u_i \|^2_{L^2(\Omega_i)} + \frac{1}{h_{ij}} \| u_i - u_j \|^2_{L^2(E_{ij})} \\ &\quad + \left(1 + \log\frac{H_j}{h_j}\right) \| \nabla u_j \|^2_{L^2(\Omega_j)}. \end{aligned} \tag{A.8}$$

  We point out that the log factor above in (A.7) can be dropped if $|E_{ij}| \asymp H_i$. See Remark 5.8 above.

- Second term in (A.5). Recall that $u_\partial = 0$, hence, $\overline{u}_{\partial j} = 0$. Then, using the notation (A.6) and (A.7) we obtain

$$\begin{aligned}(\overline{u}_j)^2 &= (\overline{u}_j - \overline{u}_{j\partial} + \overline{u}_{j\partial} - \overline{u}_{\partial j})^2 \\ &\preceq \left(1 + \log \frac{H_j}{h_j}\right) \|\nabla u_j\|^2_{L^2(\Omega_j)} + \frac{1}{h_{j\partial}} \|u_j - u_\partial\|^2_{L^2(E_{j\partial})}. \end{aligned} \tag{A.9}$$

A.2. **Bound of I in (A.2) for boundary substructures.** Let $i \in \mathcal{N}_B$ and recall equation (A.3). We obtain

$$\begin{aligned}\| \nabla u_i^{(0)} \|^2_{L^2(\Omega_i)} &\preceq \sum_{\delta_{ij} \subset \partial\Omega_i,\ j \in \mathcal{N}_I} (\overline{u}_j)^2 \, \| \Theta_i^{(j)} \|^2_{H^{1/2}_{00}(\delta_{ij})} \\ &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \sum_{\delta_{ij} \subset \partial\Omega_i,\ j \in \mathcal{N}_I} \overline{u}_j^2. \end{aligned} \tag{A.10}$$

Here again, the log factor above in (A.10) can be dropped if $|E_{j\partial}| \asymp H_j$. See also Remark 5.8.

In order to finish the estimation of term I in (A.2) when $i \in \mathcal{N}_B$ we only need to bound the right hand side of (A.10). To estimate the term $\overline{u}_j^2$ with $j \in \mathcal{N}_I$ we use

$$(\overline{u}_j)^2 \preceq \left\{(\overline{u}_j - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_i)^2 + (\overline{u}_i - \overline{u}_{i\partial} + \overline{u}_{i\partial} - \overline{u}_{\partial i})^2\right\}$$

and then apply the same arguments given above. Substituting (A.8) and (A.9) into (A.5) and recalling that $\rho_i \asymp \rho_{ij} \preceq \rho_j$ and $h_i \asymp h_{ij} \preceq h_j$ on every slave side δ_{ij}, we obtain

$$\begin{aligned}\rho_i \| \nabla u^{(0)} \|^2_{L^2(\Omega_i)} &\preceq \left(1 + \log \frac{H}{h}\right)^2 \Big\{\rho_i \| \nabla u_i \|^2_{L^2(\Omega_i)} \\ + \sum_{E_{ij} \subset \Omega_i} \Big\{\rho_j \| \nabla u_j \|^2_{L^2(\Omega_j)} &+ \frac{\rho_{ij}}{h_{ij}} \| u_i - u_j \|^2_{L^2(E_{ij})} + \frac{\rho_{j\partial}}{h_{j\partial}} \| u_j - u_\partial \|^2_{L^2(E_{j\partial})} \Big\}\Big\}. \end{aligned} \tag{A.11}$$

A.3. **Bound of term II in (A.2).** It remains to estimate the second term of (A.2). Observe that the estimate is obvious for $E_{i\partial}$ since $u^{(0)} = 0$ on $E_{\partial i}$ and $E_{i\partial}$ when $i \in \mathcal{N}_B$. We consider separately the case $i, j \in \mathcal{N}_I$ (in Section A.3.1) and the case $i \in \mathcal{N}_I$ and $j \in \mathcal{N}_B$ (in Section A.3.2).

A.3.1. *The case of two interior substructures* $i, j \in \mathcal{N}_I$. We further consider separately the cases when E_{ij} is a master and a slave side.

- Assume that $E_{ij} = \gamma_{ij}$ is a master side. We have that on E_{ij}

$$\begin{aligned}u_i^{(0)} - u_j^{(0)} &= \overline{u}_i \Theta_i^{(i)} - \left(\overline{u}_j \Theta_j^{(j)} + \overline{u}_i \Theta_j^{(i)}\right) \\ &= (\overline{u}_i - \overline{u}_j)\Theta_j^{(j)}. \end{aligned} \tag{A.12}$$

Hence,

$$\frac{1}{h_{ij}} \| u_i^{(0)} - u_j^{(0)} \|^2_{L^2(E_{ij})} = \frac{1}{h_{ij}} (\overline{u}_i - \overline{u}_j)^2 \, \| \Theta_j^{(j)} \|^2_{L^2(E_{ij})} \preceq (\overline{u}_i - \overline{u}_j)^2$$

where we have used that $h_j \asymp h_{ij} \preceq h_i$ and

$$\| \Theta_j^{(j)} \|^2_{L^2(E_{ij})} \preceq h_j \tag{A.13}$$

since $\Theta_j^{(j)}$ vanishes on $\overset{\circ}{E}_{jih_j}$. Using (A.8) and $\rho_j \asymp \rho_{ij} \preceq \rho_i$ we obtain

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel^2_{L^2(E_{ij})} &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} \\ &+ \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(E_{ij})} \\ &+ \left(1 + \log \frac{H_j}{h_j}\right) \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} . \end{aligned} \tag{A.14}$$

- Assume that $E_{ij} = \delta_{ij}$ is a slave side. In this case we have that on E_{ij} (see (A.12)),

$$u_i^{(0)} - u_j^{(0)} = \overline{u}_i \Theta_i^{(i)} + \overline{u}_j \Theta_i^{(j)} - \overline{u}_j \Theta_j^{(j)} = (\overline{u}_i - \overline{u}_j)\Theta_i^{(i)},$$

therefore, we get

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{0)} \parallel^2_{L^2(E_{ij})} &= \frac{\rho_{ij}}{h_{ij}} (\overline{u}_i - \overline{u}_j)^2 \parallel \Theta_i^{(i)} \parallel^2_{L^2(E_{ij})} \\ &\preceq \rho_{ij}(\overline{u}_i - \overline{u}_j)^2 \preceq \left(1 + \log \frac{H_i}{h_i}\right) \rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} \\ &+ \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(E_{ij})} + \left(1 + \log \frac{H_j}{h_j}\right) \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} \end{aligned} \tag{A.15}$$

in view of (A.13) for $\delta_{ij} \subset \partial\Omega_i$ and (A.8).

A.3.2. *The case of an interior substructure $i \in \mathcal{N}_I$ and a boundary substructure $j \in \mathcal{N}_B$.* Since $i \in \mathcal{N}_I$ then $u_j^{(0)}$ vanishes on E_{ji}. If $E_{ij} = \delta_{ij}$ is a slave side or $E_{ij} = \gamma_{ij}$ is a master side then

$$\frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel^2_{L^2(E_{ij})} \preceq \rho_{ij}(\overline{u}_i)^2$$

and using that

$$\overline{u}_i^2 \preceq (\overline{u}_i - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_j)^2 + (\overline{u}_j - \overline{u}_{j\partial} + \overline{u}_{j\partial} - \overline{u}_{\partial j})^2$$

and the same arguments given before, the estimate follows. The case $i \in \mathcal{N}_B$ and $j \in \mathcal{N}_I$ follows from the previous case.

Substituting (A.11), (A.14) and (A.15) into (A.1) we get (5.22).

Appendix B. Proof of Lemma 5.4

To prove (5.23) we can replace the terms $\widehat{a}_h(I_i u^{(i)}, I_i u^{(i)})$ and $b_i(u^{(i)}, u^{(i)})$ by $d_h(\mathcal{H}_i I_i u^{(i)}, \mathcal{H} I_i u^{(i)})$ and $d_i(\mathcal{H}_i u^{(i)}, \mathcal{H}_i u^{(i)})$, respectively; see Lemma 2.2 and Lemma 3.1.

In order to simplify notation, all the functions are considered as harmonic extensions in the $\mathcal{H}$ sense. Hence, we denote $\mathcal{H} I_i u^{(i)}$ by $D^{(i)} u^{(i)}$ and $\mathcal{H} u^{(i)}$ by $u^{(i)}$ and let $u^{(i)} = \{u_l^{(i)}\}_{l \in \#(i)} \in V_i$. Using (2.8), (2.9) and (4.7) we obtain

$$\begin{aligned} d_h(D^{(i)} u^{(i)}, D^{(i)} u^{(i)}) &= d_i(D^{(i)} u^{(i)}, D^{(i)} u^{(i)}) \\ &\quad + \sum_j d_j(D^{(i)} u^{(i)}, D^{(i)} u^{(i)}) \\ &= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{B.1}$$

Here, in term II, the sum is taken over all j such that Ω_j has common edges or part of edges with Ω_i. In oder to finish the proof of the Lemma we bound term I and term II in (B.1) separately in Section B.1 and Section B.2.

B.1. **Bound of term I in (B.1).** From the definition of d_i in (2.8) we write

$$\begin{aligned} & d_i(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \\ & = \rho_i \|\nabla D_i^{(i)} u_i^{(i)}\|^2_{L^2(\Omega_i)} + \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \parallel D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)} \parallel^2_{L^2(E_{ij})} . \end{aligned} \tag{B.2}$$

To get the bound of term I in (B.1) we estimate the two terms in (B.2) above.

B.1.1. *The first term of (B.2).* We note that

$$\begin{aligned} & \rho_i \parallel \nabla D_i^{(i)} u_i^{(i)} \parallel^2_{L^2(\Omega_i)} \\ & \leq 2\rho_i \left\{ \parallel \nabla (D_i^{(i)}u_i^{(i)} - u_i^{(i)}) \parallel^2_{L^2(\Omega_i)} + \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)} \right\} \end{aligned} \tag{B.3}$$

and observe that from the definition of $D_i^{(i)}$ in (4.5) and (4.6) we have

$$\rho_i \parallel \nabla (D_i^{(i)}u_i^{(i)} - u_i^{(i)}) \parallel^2_{L^2(\Omega_i)} \leq C \sum_{\delta_{ij}\subset\partial\Omega_i} \rho_i \parallel \widetilde{u}_i^{(i)} \parallel^2_{H^{1/2}_{00}(\delta_{ij})} . \tag{B.4}$$

Here, $\widetilde{u}_i^{(i)} = u_i^{(i)}$ at the nodal points of $\overset{\circ}{\delta}_{ijh_i}$, and $\widetilde{u}_i^{(i)} = 0$ on $\partial^e\delta_{ijh_i}$ and at the remaining nodes of $\partial\Omega_{ih_i}$. Note that the support of $\widetilde{u}_i^{(i)}$ is contained in δ_{ij}. Also recall that δ_{ij} denotes E_{ij} when E_{ij} is a slave side.

We only need a bound for the right hand side of (B.4) above. To obtain the desired estimate we analyze the two cases: $i \in \mathcal{N}_I$ and $i \in \mathcal{N}_B$.

- Assume $i \in \mathcal{N}_I$. In this case the local function $u_i^{(i)}$ has zero average value on $\partial\Omega_i$, hence, we can bound the $H^{1/2}_{00}-$norm of $\widetilde{u}_i^{(i)}$ by (see for example [33]),
$$\rho_i \parallel \widetilde{u}_i^{(i)} \parallel^2_{H^{1/2}_{00}(\delta_{ij})} \preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i |u_i^{(i)}|^2_{H^1(\Omega_i)}.$$
- Assume $i \in \mathcal{N}_B$. In this case we have
$$\begin{aligned} \rho_i\|\widetilde{u}^{(i)}\|^2_{H^{1/2}_{00}(\delta_{ij})} & \preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} \right\} \\ & \preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})} \right\} \end{aligned} \tag{B.5}$$
where $E_{i\partial} \subset \partial\Omega$. To get the inequality in (B.5) we have used the following estimate
$$\begin{aligned} \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} & \leq \frac{2}{H_i^2}\left\{ \|u_i^{(i)} - \overline{u}_{i\partial}^{(i)}\|^2_{L^2(\Omega_i)} + \|\overline{u}_{i\partial}^{(i)}\|^2_{L^2(\Omega_i)} \right\} \\ & \preceq |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})} \end{aligned} \tag{B.6}$$
where $\overline{u}_{i\partial}^{(i)} = \int_{E_{i\partial}} u_i^{(i)} ds / |E_{i\partial}|$ and we have used the assumption $|E_{i\partial}| \asymp H_i$. See Remark B.1 below for a discussion on this assumption.

Using the estimates (B.4) and (B.5) in (B.3) we get the estimate for the first term of (B.2),

$$\rho_i \|\nabla D_i^{(i)} u_i^{(i)}\|^2_{L^2(\Omega_i)} \preceq \left(1+\log\frac{H_i}{h_i}\right)^2 \rho_i \Big\{ \|\nabla u_i^{(i)}\|^2_{L^2(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})}\Big\}. \tag{B.7}$$

Remark B.1. It is important to note that we have used the assumption $|E_{i\partial}| \asymp H_i$ in order to avoid an extra log factor in the inequality (B.6). In case this assumption is not satisfied, the coarse basis function θ_i must be added to the coarse problem to obtain the estimate with $(1+\log(H_i/h_i))^2$ factor. We point out that the coarse space $V_{0,I\cup B}$ defined in (5.25) includes automatically such functions; see Theorem 5.7 where we analyze the method that uses the coarse space $V_{0,I\cup B}$. For the proof of Theorem 5.2 we do not assume that the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i.

B.1.2. *The second terms of (B.2).* First note that the estimate is straightforward for boundary edges $E_{i\partial}$ since by definition $u_\partial^{(i)} = 0$ and $D_i^{(i)} = 1$ on $E_{i\partial}$. We need to analyze separately the cases of slave ($E_{ij} = \delta_{ij}$) and master side ($E_{ij} = \gamma_{ij}$).

- Assume $E_{ij} = \delta_{ij}$ is a slave side. From (4.5) and (4.6) we have

  $$\| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|^2_{L^2(\delta_{ij})} \preceq h_i \max_{\delta_{ij}} |u_i^{(i)}|^2,$$

  and recalling that $\rho_i \asymp \rho_{ij} \preceq \rho_j$ and $h_i \asymp h_{ij} \preceq h_j$ we obtain

  $$\frac{\rho_{ij}}{h_{ij}} \| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|^2_{L^2(\delta_{ij})} \preceq \rho_i \max_{\delta_{ij}} |u_i^{(i)}|^2 \preceq \left(1+\log\frac{H_i}{h_i}\right) \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} \right\}. \tag{B.8}$$

  To estimate of the second term of the right-hand side in (B.8) we use a Poincaré inequality (recall $u_i^{(i)}$ has zero average value on $\partial\Omega_i$) when $i \in \mathcal{N}_I$, and we use the inequality (B.6) when $i \in \mathcal{N}_B$. Thus

  $$\frac{\rho_{ij}}{h_{ij}} \|D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)}\|^2_{L^2(\delta_{ij})} \preceq (1+\log\frac{H_i}{h_i})\ \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})} \right\}. \tag{B.9}$$

- Assume $E_{ij} = \gamma_{ij}$ is a master side. Remember that on a master side, $h_j \asymp h_{ij} \preceq h_i$ and $\rho_j \asymp \rho_{ij} \preceq \rho_i$. We have

  $$\| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|_{L^2(\gamma_{ij})} \le \| u_i^{(i)} - u_j^{(i)} \|_{L^2(\gamma_{ij})} + \| z_j^{(i)} \|_{L^2(E_{ji})}, \tag{B.10}$$

  where

  $$z_j^{(i)} = \sum_{x_k^j \in \partial^e E_{jih_j}} u_j^{(i)}(x_k^j)\varphi_k^j.$$

  Here, φ_k^j are the nodal basis functions on E_{ji,h_j} corresponding to the nodes x_k^j on $\partial^e E_{ji,h_j}$. Let us denote the support of $z_j^{(i)}$ by $S_{z_j^{(i)}}$ on E_{ji} and see that $|S_{z_j^{(i)}}| \preceq h_j$. We have

$$\| z_j^{(i)} \|_{L^2(S_{z_j^{(i)}})} \preceq \| u_j^{(i)} \|_{L^2(S_{z_j^{(i)}})} \preceq \|u_j^{(i)} - u_i^{(i)}\|^2_{L^2(\gamma_{ij})} + \|u_i^{(i)}\|^2_{L^2(S_{z_j^{(i)}})}. \tag{B.11}$$

The second term of the right-hand side of (B.11) can be estimated by

$$\|u_i^{(i)}\|^2_{L^2(S_{z_j^{(i)}})} \preceq Ch_j \max_{E_{ij}} |u_i^{(i)}|^2$$
$$\preceq h_i \left(1+\log\frac{H_i}{h_i}\right)\left\{|u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})}\right\}, \tag{B.12}$$

where we have used a Poincaré inequality for $i \in \mathcal{N}_I$ and the estimate (B.6) for $i \in \mathcal{N}_B$. Using (B.11) and (B.12) in (B.10) we get

$$\frac{\rho_{ij}}{h_{ij}}\|D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)}\|^2_{L^2(E_{ij})} \preceq \tag{B.13}$$
$$\frac{\rho_{ij}}{h_{ij}}\|u_j^{(i)} - u_i^{(i)}\|^2_{L^2(E_{ij})} + \left(1+\log\frac{H_i}{h_i}\right)\rho_i\left\{\|\nabla u_i^{(i)}\|^2_{L^2(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(E_{i\partial})}\right\}.$$

We now use the estimates (B.7), (B.8) and (B.13) in (B.2) and Lemma 2.2 to obtain an estimate for second terms of (B.2),

$$d_i(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \preceq \left(1+\log\frac{H_i}{h_i}\right)^2 b_i(u^{(i)},u^{(i)}). \tag{B.14}$$

B.2. **Bound of term II in (B.1).** We now estimate the second term of (B.1) by bounding $d_j(D^{(i)}u^{(i)}, D^{(i)}u^{(i)})$ by $b_i(u^{(i)},u^{(i)})$. For $u^{(i)} = \{u_j^{(i)}\} \in V_i$ we have

$$d_j(D^{(i)}u^{(i)}, D^{(i)}u^{(i)})$$
$$= \rho_j \parallel \nabla D_j^{(i)}u_j^{(i)} \parallel^2_{L^2(\Omega_j)} + \frac{\delta}{l_{ij}}\frac{\rho_{ij}}{h_{ij}}\int_{E_{ij}} (D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)})^2 dx. \tag{B.15}$$

We only need to estimate the first term of (B.15) since the second term has been already estimated; see (B.8) and (B.13).

If $E_{ij} = \delta_{ij}$ is a slave side then $D_j^{(i)}$ vanishes, and so $\parallel \nabla D_j^{(i)}u_j^{(i)} \parallel^2_{L^2(\Omega_j)}$ vanishes as well.

We now estimate the case where $E_{ij} = \gamma_{ij}$ is a master side. On E_{ji} we decompose $u_j^{(i)} = w_j^{(i)} + \sum_{x_k^j \in \partial^e E_{jih_j}} u_j^{(i)}(x_k^j)\varphi_k^j$, where $w_j^{(i)} = D_j^{(i)}u_j^{(i)}$, i.e., $w_j^{(i)}$ equals $u_j^{(i)}$ at the nodes in $\overset{\circ}{E}_{jih_j}$ and zero at the nodes in $\partial^e E_{jih_j}$. Note that the support of $w_j^{(i)}$ belongs to E_{ji}. We have

$$\parallel \nabla w_j^{(i)} \parallel^2_{L^2(\Omega_j)} \preceq \parallel w_j^{(i)} \parallel^2_{H^{1/2}_{00}(E_{ji})}$$
$$= \{|w_j^{(i)}|^2_{H^{1/2}(E_{ji})} + \int_{E_{ji}} \frac{(w_j^{(i)})^2}{\mathrm{dist}(s,\partial E_{ji})}\, ds\}. \tag{B.16}$$

We now estimate the first term of (B.16). Let Q_j be the L_2- projection on the h_j-triangulation of E_{ji}. Then

$$|w_j^{(i)}|^2_{H^{1/2}(E_{ji})} \leq 2\{|w_j^{(i)} - Q_j u_i^{(i)}|^2_{H^{1/2}(E_{ji})} + |Q_j u_i^{(i)}|^2_{H^{1/2}(E_{ji})}\} \tag{B.17}$$
$$\preceq \frac{1}{h_j}\parallel w_j^{(i)} - u_i^{(i)} \parallel^2_{L^2(E_{ji})} + \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)}$$

and

$$\begin{aligned}&\| w_j^{(i)} - u_i^{(i)} \|^2_{L^2(E_{ji})}\\ &\leq 2\{\| u_j^{(i)} - u_i^{(i)} \|^2_{L^2(E_{ji})} + \| \sum_{x_k^j \in \partial^e E_{ji,h_j}} u_j^{(i)}(x_k^j)\varphi_k^j \|^2_{L^2(E_{ji})}\}\end{aligned} \tag{B.18}$$

where the second term of (B.18) can be bounded as before, see (B.10)-(B.12).

It remains to estimate the second term of (B.16). In order to simplify the arguments, we take E_{ij} as the interval $[0,H]$. Note that

$$\int_{E_{ji}} \frac{(w_j^{(i)})^2}{\mathrm{dist}(s,\partial E_{ji})} ds \preceq \int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds + \int_{H/2}^{H} \frac{(w_j^{(i)})^2}{(H-s)}. \tag{B.19}$$

Let us estimate the first term in the right-hand side of (B.19). Let A be the most far left node of $\mathring{E}_{jih_j}$ in $[0,H/2]$ and note the size of the interval of $[0,A]$ is $O(h_j)$. We have

$$\begin{aligned}
\int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds &= \int_0^{A} \frac{(w_j^{(i)})^2}{s} ds + \int_A^{H/2} \frac{(u_j^{(i)})^2}{s} ds \\
&\preceq (u_j^{(i)}(A))^2 + \int_A^{H/2} \frac{(u_i^{(i)} - u_j^{(i)})^2}{s} ds + \int_A^{H/2} \frac{(u_i^{(i)})^2}{s} ds \\
&\preceq (u_j^{(i)}(A))^2 + \frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|^2_{L^2(E_{ji})} \\
&\quad + \left(1 + \log \frac{H_j}{h_j}\right) \max_{E_{ij}} |u_i^{(i)}|^2 \\
&\preceq \frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|^2_{L^2(E_{ij})} \\
&+ \left(1 + \log \frac{H_i}{h_i}\right)\left(1 + \log \frac{H_j}{h_j}\right) \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i} \|u_i^{(i)}\|^2_{L^2(E_{i\partial})} \right\}
\end{aligned}$$

where $(u_j^{(i)}(A))^2$ has been estimated using (B.11) and (B.12). The second term of (B.19) is estimated similarly. Substituting these estimates in (B.19) we get

$$\begin{aligned}&\int_{E_{ji}} \frac{(w_j^{(i)})^2}{\mathrm{dist}(s,\partial E_{ji})} ds\\ &\preceq \left(1 + \log \frac{H}{h}\right)^2 \{\| \nabla u_i^{(i)} \|^2_{L^2(\Omega_i)} + \frac{1}{h_j} \| u_i^{(i)} - u_j^{(i)} \|^2_{L^2(E_{ij})} + \frac{1}{h_i} \|u_i^{(i)}\|^2_{L^2(E_{i\partial})}\}.\end{aligned} \tag{B.20}$$

Substituting (B.17) together with the estimate for (B.18) and (B.20) into (B.16), and then substituting this resulting estimate with (B.8) and (B.13) into (B.15), and using Lemma 2.2, we get

$$d_j(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \preceq \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}). \tag{B.21}$$

Using (B.14) and (B.21) in (B.1), we get

$$d_h(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \preceq \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}).$$

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