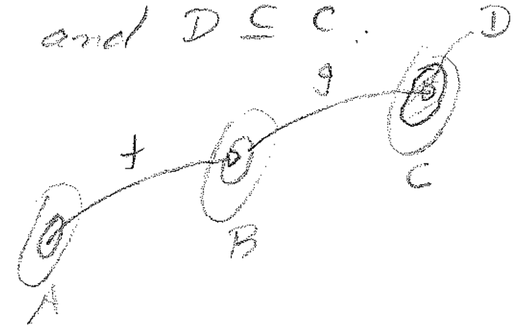
HW 4.

(Ex 4.3) 4   Let $f : A \to B$, $g : B \to C$ and $D \subseteq C$.

Then $(g \circ f)^{-1}(D) = f^{-1}(g^{-1}(D))$.



Pf

Let $x \in (g \circ f)^{-1}(D)$. Then $\exists\, z \in D$ where

$(g \circ f)(x) = z$ and define $y = f(x)$. Therefore,

$$y \in g^{-1}(D) \text{ since } g(y) = g(f(x)) = z \in D.$$

Also $x \in f^{-1}(\{y\}) \subseteq f^{-1}(g^{-1}(D))$

since

$$\{\tilde{x} \in A \mid f(x) = y\} \subseteq \{\tilde{x} \in A \mid f(x) = \tilde{y} \text{ for some } \tilde{y} \in g^{-1}(D)\}$$

which includes y.

Let $x \in f^{-1}(g^{-1}(D))$. Define $y = f(x)$ and we have $y \in g^{-1}(D)$.

Hence, $\exists\, z \in D$ where $g(y) = z$

$$g(y) = g(f(x)) = (g \circ f)(x).$$

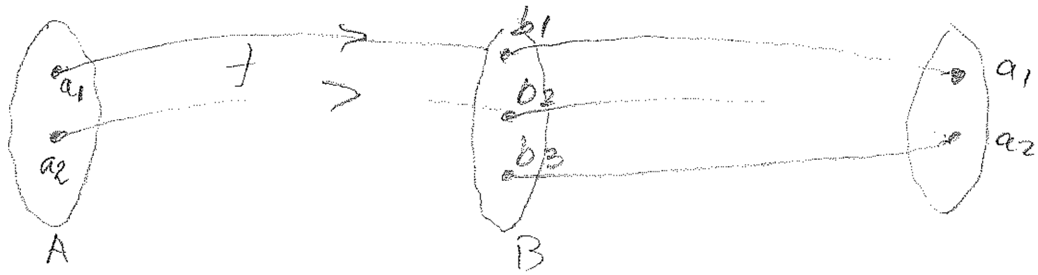
This means that

$$x \in (g \circ f)^{-1}(D).$$

(Ex 4.3) 4

False Proposition : $f: A \to B$, $g: B \to A$.

If $g \circ f = I_A$, then g is the inverse function of A.



$f = \{(a_1, b_1), (a_2, b_2)\}$

$g = \{(b_1, a_1), (b_2, a_2), (b_3, a_2)\}$

$g \circ f = \{(a_1, a_1), (a_2, a_2)\} = I_A$

$f \circ g = \{(b_1, b_1), (b_2, b_2), (b_3, b_2)\} \neq I_B$.

(Ex 4.5) 18 Let $f: X \to Y$. Then f is 1-1 $\iff$ $f^{-1}(f(A)) = A$ $\forall A \subseteq X$

Pf Assume f is 1-1 and $A \subseteq X$.

$\Rightarrow$ Let $x \in A$. Let $y \in f(A)$ defined by $y = f(x)$.

Then $x \in f^{-1}(\{y\}) \subseteq f^{-1}(f(A))$.

$\Leftarrow$ Let $x \in f^{-1}(f(A))$. $\exists\, y \in f(A)$ such that $f(x) = y$.

Since $y \in f(A)$, $\exists\, \tilde{x} \in A$ where $f(\tilde{x}) = y$.

Since f is 1-1 $\Rightarrow$ $x = \tilde{x} \in A$.

(Ex 5.4) 4 $\forall m \in \mathbb{N}$ and $x \in \mathbb{R} - \{-1\}$

$$\sum_{k=0}^{2m-1} (-1)^{k+1} x^{k} = \frac{x^{2m} - 1}{x+1}$$

Pf By induction

Base $m=1$ $\displaystyle\sum_{k=0}^{1} (-1)^{k+1} x^{k} = -1 + x = x - 1 \Big\} \ \frac{x^{2}-1}{x+1} = x - 1$

Assume true for m.

$$\sum_{k=0}^{2(m+1)-1} (-1)^{k+1} x^{k} = \sum_{k=0}^{2m-1} (-1)^{k+1} x^{k} + (-1)^{2m+1} x^{2m} + (-1)^{2m+2} x^{2m+1}$$

$$= \frac{x^{2m} - 1}{x+1} - x^{2m} + x^{2m+1} = \frac{\cancel{x^{2m}} - 1 - \cancel{x^{2m+1}} - \cancel{x^{2m}} + x^{2m+2} + \cancel{x^{2m+1}}}{x+1}$$

$$= \frac{x^{2m+2} - 1}{x+1}$$

Therefore we showed that

$$\sum_{k=0}^{2(m+1)-1} (-1)^{k+1} x^{k} = \frac{x^{2(m+1)} - 1}{x+1}$$

(Ex 5.5) 8 Let $f: A \to B$. If there exists $g: B \to A$ onto and $f \circ g = I_B$

$$\Downarrow$$

f is 1-1

Pf

Let $y \in B$. Then $y = I_B y = (f \circ g)(y) = f(g(y))$. Denote $x = g(y)$. Hence $f(x) = y$. To show that f is 1-1 we have to show that if $\tilde{x} \neq x$ then $f(\tilde{x}) \neq y$.

Indeed, let $\tilde{x} \neq x$. Since g is onto, $\exists \tilde{y} \in B$ such that $g(\tilde{y}) = \tilde{x}$. We note however that $\tilde{y} \neq y$ since $g(y) = x \neq \tilde{x}$. Therefore $\tilde{y} = I_B \tilde{y} = (f \circ g)(\tilde{y}) = f(\tilde{x})$, and we have show that if $\tilde{x} \neq x$ then $f(\tilde{x}) = \tilde{y} \neq y$.

(Ex 5.8) 4 If $B \subseteq A \Rightarrow P(B) \subseteq P(A)$

Pf Let $S \in P(B)$. Then $S \subseteq B \subseteq A \Rightarrow S \subseteq A \Rightarrow S \in P(A)$.

(Ex 5.9) 4 $\mathcal{K} = \{\{7, 11\}, \{7\}\}$ Find $\theta^{-1}(\mathcal{K})$ (5)

$$\mathcal{D} = \{ G \in P(A) \mid 7 \notin G \}$$
$$= \{ \{11\}, \{13\}, \{11, 13\}, \phi \}$$

$$\theta : \mathcal{D} \to \mathcal{H}$$
$$\theta(G) = G \cup \{7\}$$

$$\mathcal{H} = \{ H \in P(A) \mid 7 \in H \}$$
$$= \{ \{7\}, \{7, 11\}, \{7, 13\}, \{7, 11, 13\} \}$$

$$\theta^{-1}(\mathcal{K}) = \{ \{11\}, \phi \}$$

(Ex 5.12) 4 $\mathbb{N} \times \mathbb{N}$ is countably infinite.

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

$$\begin{array}{lccccc}
\frac{2 \times 1}{2} & \to & 1 & 2 & 4 & 7 \\
\frac{3 \times 2}{2} & \to & 3 & 5 & 8 & \\
\frac{4 \times 3}{2} & \to & 6 & 9 & & \\
\frac{5 \times 4}{2} & \to & 10 & & &
\end{array}$$

1 $\to$ (1,1)
2 $\to$ (1,2)
3 $\to$ (2,1)
4 $\to$ (1,3)
5 $\to$ (2,2)
$\vdots$
etc

how to write $f(n)$?

Given n, find the smallest $K \in \mathbb{N}$ such that

$\frac{(K+1)K}{2} \geq n$. Let $m = \frac{(K+1)K}{2} - n$

$f(n) = (K-m, 1+m)$

Ex. Let $n = 9 \Rightarrow K = 4 \Rightarrow m = \frac{5 \times 4}{2} - 9 = 1$

$f(9) = (4-1, 1+1) = (3, 2)$

(Ex 5.14) 6 The set $\mathbb{R} - \mathbb{Q}$ of irrational numbers is uncountable.

Pf. By contradiction, suppose $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ is countable infinite (it is not finite since $\frac{1}{\sqrt{3}} + \frac{1}{2K}$, $\forall K \in \mathbb{N}$ is irrational)

Then exists

$f : \mathbb{N} \to \mathbb{I}$ (1-1 and onto).

We know that $\mathbb{Q}$ is countable infinite, so exists

$g : \mathbb{N} \to \mathbb{Q}$ (1-1 and onto).

Define

$$h : \mathbb{N} \to \mathbb{R} \text{ as } \begin{cases} h(n) = f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ h(n) = g\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

Easy to check that h is 1-1 and onto over $\mathbb{R}$.
Contradiction since $\mathbb{R}$ is uncountable. So $\mathbb{I}$ is uncountable.