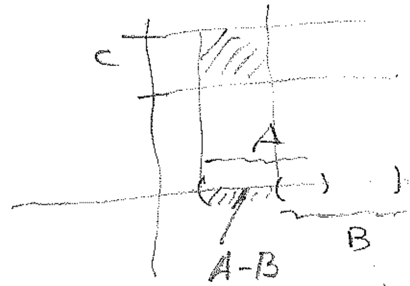
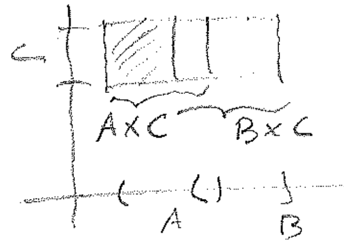
HW 3

(Ex 3.1) 6

Thm: $\forall A, B, C$, $(A - B) \times C = (A \times C) - (B \times C)$





($\Rightarrow$)

Pf. Let $(x,y) \in (A-B) \times C$, that is, $x \in A - B$ and $y \in C$.

$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } y \in C$.

$\Rightarrow (x \in A \text{ and } x \in C) \text{ and } (x \notin B \text{ and } y \in C)$

$\Rightarrow (x,y) \in (A \times C) \text{ and } (x,y) \notin (B \times C)$

$\Rightarrow (x,y) \in (A \times C) - (B \times C)$

($\Leftarrow$)

Let $(x,y) \in (A \times C) - (B \times C)$, that is, $\Rightarrow$

$(x,y) \in (A \times C)$ and $(x,y) \notin (B \times C)$.

$(x \in A \text{ and } y \in C)$ and $((x \in B \text{ and } y \notin C) \text{ or } (x \notin B \text{ and } y \in C)$ or $(x \notin B \text{ and } y \notin C))$

Since $y \in C$ $\Rightarrow$

$(x \in A \text{ and } y \in C) \text{ and } (x \notin B \text{ and } y \in C) \Rightarrow$

$(x \in A \text{ and } x \notin B) \text{ and } y \in C \Rightarrow$

$x \in A - B$ and $y \in C \Rightarrow (x,y) \in (A-B) \times C$

(Ex 3.2) 2

1. $Dom(r) = \mathbb{R}$, $Im(r) = \mathbb{R}$
2. $Dom(r) = \mathbb{R}$, $Im(r) = \mathbb{R}$
3. $Dom(r) = \mathbb{R}$, $Im(r) = \{x \in \mathbb{R} \mid x \geq 0\} = \mathbb{R}^+ \cup \{0\}$
4. $Dom(r) = \mathbb{N}$, $Im(r) = \{3, 5, 7, \ldots\} =$ odd numbers $- \{1\}$
5. $Dom(r) = \mathbb{Z}$, $Im(r) = \mathbb{Z}$
6. $Dom(r) = \mathbb{Z}$, $Im(r) = \mathbb{Z}$
7. $Dom(r) = \mathbb{R}^2$, $Im(r) = \mathbb{R}^2$
8. $Dom(r) = \mathbb{N}$, $Im(r) = \mathbb{N}$
9. $Dom(r) = \mathbb{Z}$, $Im(r) = \mathbb{Z}$.

(Ex 3.3) 10 r on $\mathbb{R}^2$ $\mid (u,v)\, r\, (x,y) \iff y - x^2 = v - u^2$

r is symmetric: if $(u,v)\, r\, (x,y) \Rightarrow y - x^2 = v - u^2 \Rightarrow v - u^2 = y - x^2$
$\Rightarrow (x,y)\, r\, (u,v)$

r is reflexive : Since $y - x^2 = y - x^2 \Rightarrow (x,y)\, r\, (x,y)$

r is transitive : if $(u,v)\, r\, (x,y)$ and $(x,y)\, r\, (s,t)$ then
$v - u^2 = y - x^2 = t - s^2 \Rightarrow (u,v)\, r\, (s,t)$

(Ex 3.4) 4 Find equivalence class of $a \equiv b \bmod 3$, $a,b \in \mathbb{Z}$

$[1] = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\} = \{m \in \mathbb{Z} \mid m = 3n + 1 \ \forall n \in \mathbb{Z}$

$[2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\} = \{m \in \mathbb{Z} \mid m = 3n + 2 \ \forall n \in \mathbb{Z}\}$

$[3] = \{\ldots, -3, 0, 3, 6, 9, 12, \ldots\} = \{m \in \mathbb{Z} \mid m = 3n + 3 \ \forall n \in \mathbb{Z}\}$

(Ex 3.5) 6 Find $A = \bigcap_{k \in \mathbb{N}} \{ m \in \mathbb{N} \mid m \geq k \}$.

$A = \{1, 2, 3, 4, \ldots\} \cap \{2, 3, 4, 5, \ldots\} \cap \{3, 4, 5, 6, \ldots\} \cap \ldots$

It is easy to see that A is empty

Motivation.

By way of contradiction, assume that $A \neq \emptyset$.

Then $\exists n \in \mathbb{N}$ such that $n \in A_k$ for all $k \in \mathbb{N}$.

$A_k = \{ m \in \mathbb{N} \mid m \geq k \} = \{k, k+1, \ldots\}$.

Clear $n \notin A_k$ for $k > n$. contradiction

(Ex 3.6) 2

$\{S_1, S_2, S_3, S_4\}$ is a partition of $A = \{13, 89, 421, 5, 2, 9, 7\}$

where $S_1 = \{13, 7\}$, $S_2 = \{89, 9, 2\}$, S_3 $\{421\}$ and $S_4 = \{5\}$.

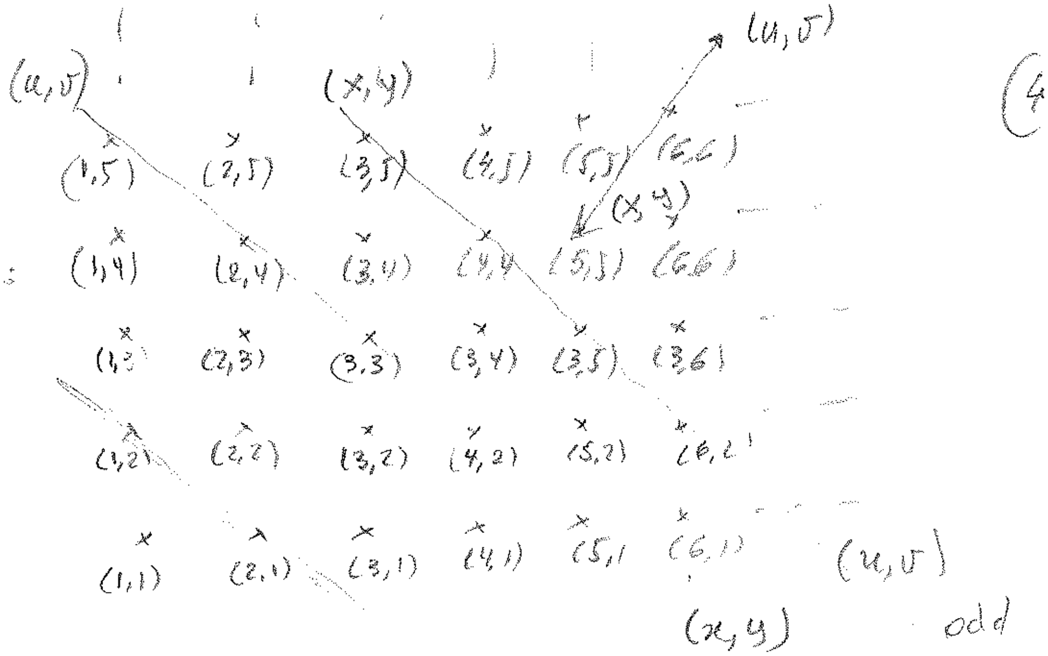
Like in Example (3.14)

$\sim = \{ (13,13), (13,7), (7,7), (7,13), (89,89), (89,9),$
$(9,89), (9,9), (89,2), (2,89), (9,2), (2,9), (2,2),$
$(421,421), (5,5) \}$

(4)

(Ex 3.7) 7

Motivation:



$(x,y)\, r\, (u,v)$

To be a partial order we need to show the following three properties:

<u>r is reflexive</u> $\to$ $(x,y)\, r\, (x,y)$ since $x+y = x+y$ and

$x \geq x$ and $y \leq y$ if $x+y$ is even and

$x \leq x$ and $y \geq y$ if $x+y$ is odd

<u>r is antisymmetric</u> $\Rightarrow$ Assume $(x,y)\, r\, (u,v)$ and $(u,v)\, r\, (x,y)$.

Then easy to see that $x+y = u+v$

Case i $x+y = u+v$ even. $\begin{cases} \Rightarrow x \geq u \text{ and } y \leq v & (\text{since } (x,y)\, r\, (u,v)) \\ \Rightarrow u \leq x \text{ and } v \leq y & (\text{since } (u,v)\, r\, (x,y)) \end{cases}$

$\Downarrow$

$x = u$ and $y = v$

Case ii) similar idea...

<u>r is transitive</u> Suppose $(s,t)\, r\, (x,y)$ and $(x,y)\, r\, (u,v)$

1) If $s+t < x+y$ and $x+y < u+v \Rightarrow s+t < u+v \Rightarrow (s,t)\, r\, (u,v)$

2) If $s+t < x+y$ and $x+y = u+v \Rightarrow s+t < u+v \Rightarrow (s,t)\, r\, (u,v)$

3) If $s+t = x+y$ and $x+y < u+v \Rightarrow s+t < u+v \Rightarrow (s,t)\, r\, (u,v)$

4) $\downarrow$ next page

4) If $s+t = x+y = u+v$ even. Then

$$\begin{matrix} s \geq x \geq u & \Rightarrow \\ t \leq y \leq v & \Rightarrow \end{matrix} \left| \begin{matrix} s \geq u \\ t \leq v \end{matrix} \right| \Rightarrow (s,t) R (u,v)$$

5) If $s+t = x+y = u+v$ odd. Then

$$\begin{matrix} s \leq x \leq u & \Rightarrow \\ t \geq y \geq v & \Rightarrow \end{matrix} \left| \begin{matrix} s \leq u \\ t \geq v \end{matrix} \right| \Rightarrow (s,t) R (u,v)$$

(Ex 4.1) 3 List all possible functions from $\{1,2,3\}$ to $\{1,2\}$

$f_1 = \{(1,1),(2,1),(3,1)\}$ $f_2 = \{(1,1),(2,1),(3,2)\}$

$f_3 = \{(1,1),(2,2),(3,1)\}$ $f_4 = \{(1,1),(2,2),(3,2)\}$

$f_5 = \{(1,2),(2,1),(3,1)\}$ $f_6 = \{(1,2),(2,1),(3,2)\}$

$f_7 = \{(1,2),(2,2),(3,1)\}$ $f_8 = \{(1,2),(2,2),(3,2)\}$

(Ex 4.2) 16 X and Y sets. Let $f: X \to Y$ and $A \subseteq Y$, $B \subseteq Y$.

Then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Pf Assume $x \in f^{-1}(A \cup B)$. Then $\exists y \in A \cup B$ such that $y = f(x)$. Then $y \in A$ or $y \in B$. Therefore $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, hence, $x \in f^{-1}(A) \cup f^{-1}(B)$.

Assume $x \in f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

Then $(\exists y_1 \in A$ where $f(x) = y_1)$ or $(\exists y_2 \in B$ where $f(x) = y_2)$.

Then $\exists y \in A$ or $\exists y \in B$ where $f(x) = y$.

Then $\exists y \in A \cup B$ where $f(x) = y \Rightarrow x \in f^{-1}(A \cup B)$.

(Ex 4.2) 20 Let $f: X \to Y$ and $A \subseteq X$. Then $A \subseteq f^{-1}(f(A))$.

Pf: Let $x \in A$, since f is a function, exists a unique $y \in Y$ where $f(x) = y$. Note that $y \in \underbrace{f(A)}_{B}$.

Hence $x \in f^{-1}(B) = f^{-1}(f(A))$. So $x \in f^{-1}f(A)$.

So, $A \subseteq f^{-1}(f(A))$.

(Ex 5.3) 3 For all $n \in \mathbb{N}$ $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

Pf: By induction

Base case $n = 1$ $\displaystyle\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$ $\qquad \frac{n}{n+1} \overset{n=1}{=} \frac{1}{2}$ ✓ OK

Assume true $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

We want to show $\displaystyle\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{n+2}$

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \left(\sum_{k=1}^{n} \frac{1}{k(k+1)}\right) + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2}$$