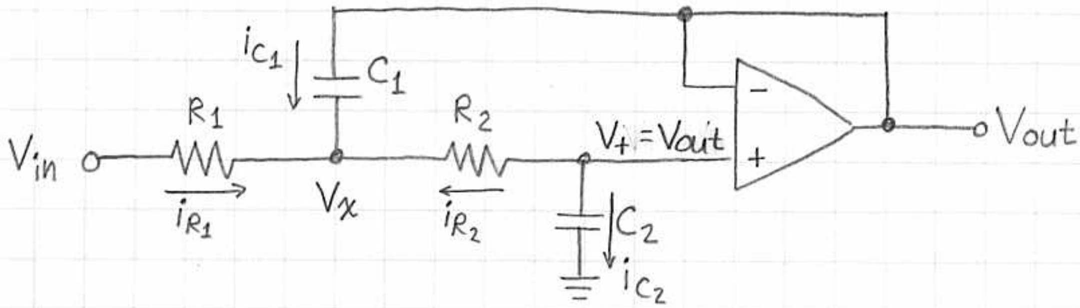
"SALLEN AND KEY" CIRCUIT

For realizing a complex pole pair in the s-plane



Analysis: KCL using node voltages to describe currents.
At V_x (using $V_+ = V_{out}$, by ideal op-amp)

$$\underbrace{\frac{(V_{in} - V_x)}{R_1}}_{i_{R_1}} + \underbrace{(V_{out} - V_x) sC_1}_{i_{C_1}} + \underbrace{\frac{(V_{out} - V_x)}{R_2}}_{i_{C_2}} = 0 \qquad [1]$$

Collecting terms and multiplying through by R_1 gives

$$V_{in} + V_{out}\left(sR_1C_1 + \frac{R_1}{R_2}\right) = V_x\left(sR_1C_1 + \frac{R_1}{R_2} + 1\right) \qquad [2]$$

KCL at V_+ gives

$$\underbrace{\frac{V_{out} - V_x}{R_2}}_{i_{R_2}} + \underbrace{V_{out}\, sC_2}_{i_{C_2}} = 0 \qquad [3]$$

Collect and multiply by R_2, giving

$$V_x = V_{out}(sR_2C_2 + 1) \qquad [4]$$

Substituting [4] into [2] to eliminate V_x and solving for V_{out}/V_{in} gives

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{1}{s^2(R_1R_2C_1C_2) + s(R_1 + R_2)C_2 + 1}} \qquad [5]$$

Which is a second order, all pole transfer function.

Design: How do we choose R_1, R_2, C_1, C_2 to place the complex pole pair where we want? We have 4 values to choose, yet the complex pole pair is completely determined by 2 numbers: the real and imaginary parts. So we have 2 free parameters among R_1, R_2, C_1, C_2.

It turns out design is simplified by letting

$$R_1 = R_2 = R \qquad [6]$$

Now 5 becomes

$$\frac{V_{out}}{V_{in}} = \frac{1}{s^2(R^2C_1C_2) + s(2RC_2) + 1}$$

The poles (roots of the denominator) are given by

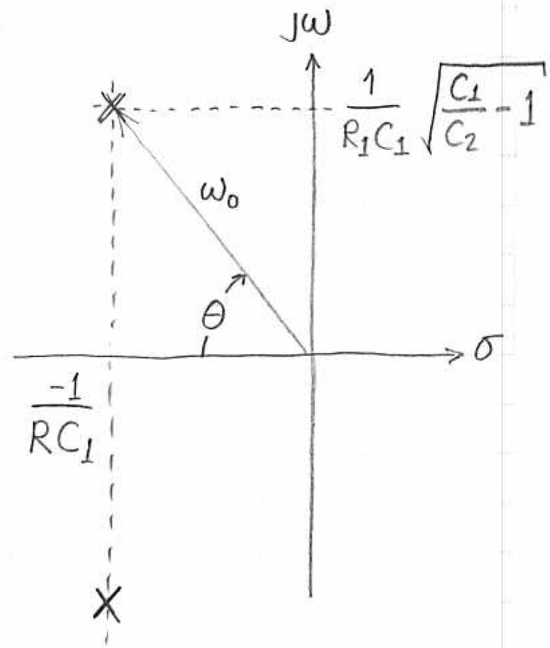
$$p_1, p_2 = \frac{-2RC_2 \pm \sqrt{4R^2C_2^2 - 4R^2C_1C_2}}{2R^2C_1C_2}$$

which reduces to

$$\boxed{p_1, p_2 = \frac{-1}{RC_1}\left(1 \pm j\sqrt{\frac{C_1}{C_2} - 1}\right)}$$

This can also be expressed in radial form. Some trigonometry gives

$$\boxed{\omega_0 = \frac{1}{R\sqrt{C_1C_2}} \qquad \tan\theta = \sqrt{\frac{C_1}{C_2} - 1}}$$



We still have R and C_1 or C_2 as free parameters, with the constraint that $C_1 > C_2$.