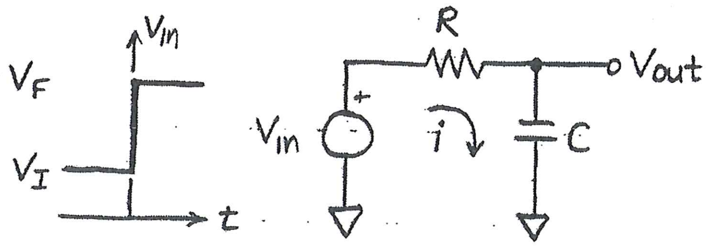
STEP RESPONSE OF FIRST ORDER SYSTEM



First order differential equation for V_{out}

$$\underbrace{V_{out} = V_{in} - iR}_{\text{KVL}} \Rightarrow \underbrace{i = C\frac{d}{dt}V_{out}}_{\text{CAPACITOR}} \Rightarrow \frac{d}{dt}V_{out} = \left(\frac{-1}{RC}\right)V_{out} + \left(\frac{1}{RC}\right)V_{in}$$

Solve differential equation; homogeneous solution: $V_{in}=0$
try solution of form $Ae^{-t/\tau}$; substitute

$$\frac{d}{dt}\left[Ae^{-t/\tau}\right] = \left(\frac{-1}{RC}\right)\left[Ae^{-t/\tau}\right]$$

$$\frac{-1}{\tau}Ae^{-t/\tau} = \frac{-1}{RC}Ae^{-t/\tau} \Rightarrow \tau = RC \Big\} \text{"TIME CONSTANT"}$$

Particular solution: when output is constant,
$t \to \infty$, $dV_{out}/dt \to 0$; try constant solution $V_{out} = B$
As $t \to \infty$, $V_{in} = V_F$

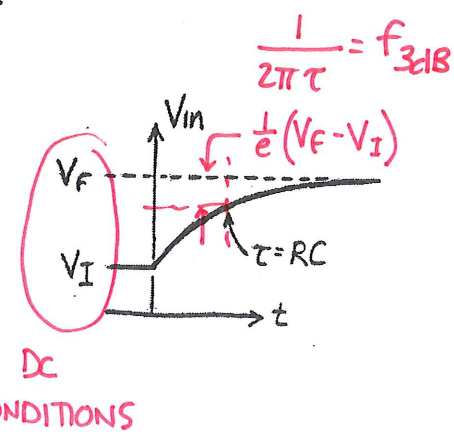
$$\left(\frac{-1}{RC}\right)B + \left(\frac{1}{RC}\right)V_F = 0 \Rightarrow B = V_F$$

Boundary condition: at $t=0$, $V_{out} = V_I$ (voltage across C cannot change instantaneously)

$$Ae^{-0/\tau} + V_F = V_I \Rightarrow A = V_I - V_F$$

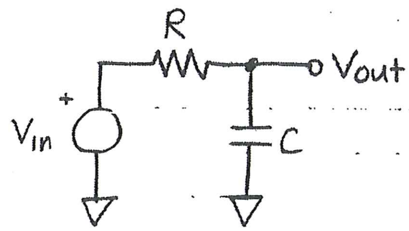
General solution:

$$\boxed{V_{out}(t) = V_F - (V_F - V_I)e^{-t/\underbrace{RC}_{\tau}}}$$



BANDWIDTH AND RISETIME

Consider the single pole network



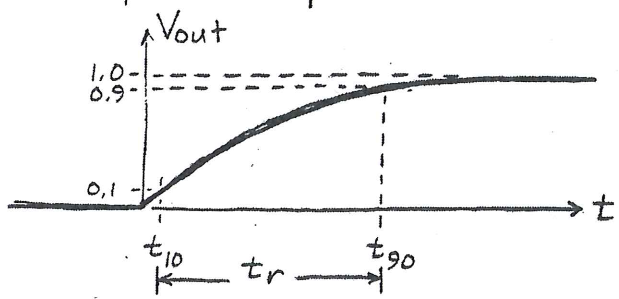
$$\frac{V_{out}}{V_{in}} = H(s) = \frac{1}{1+sRC} = \frac{1}{1+s\tau}$$

Letting $s = j\omega$, we have

$$H(j\omega) = \frac{1}{1+j\omega\tau} = \frac{1}{1+j\left(\frac{\omega}{\omega_0}\right)}$$

where $\omega_0 = 1/\tau$ is the 3dB bandwidth, in radians/sec

The "rise time" t_r is defined as the time required for the output to rise from 10% to 90% of the output step:



$BW = 8.5\ \text{KHz}$

$t_r = \frac{0.35}{8.5\text{kHz}} = 41\mu\text{sec}$ ✓

From general step response

$$\left.\begin{aligned} 0.1 = 1.0 - e^{-t_{10}/\tau} &\Rightarrow t_{10} = -\tau \ln(0.9) \\ 0.9 = 1.0 - e^{-t_{90}/\tau} &\Rightarrow t_{90} = -\tau \ln(0.1) \end{aligned}\right\} t_r = t_{90} - t_{10} = \tau \ln(9)$$

Or, $\boxed{t_r = 2.2\tau}$

Since $\tau = 1/\omega_0$, and $f_0 = \omega_0/2\pi = BW$ (3 dB bandwidth in Hz)

$$t_r = 2.2\frac{1}{\omega_0} = \frac{1}{BW}\,\frac{2.2}{2\pi} \Rightarrow \boxed{BW \cdot t_r = 0.35}$$

Approximately valid for any linear system with poles near real axis (little overshoot in step response)