Doing University Mathematics

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Abstract

This is an essay on what I see as the elements of university mathematics. In addition to teaching you the basics of algebra, discrete math, or number theory (or whatever), I hope to instill in you a sense of what mathematics is and how to go about doing it on your own. Perhaps what we are after here is a sense of "mathematical maturity". As mathematics students, you should strive to enable yourselves to learn independently and to think critically.

If nothing else, these notes will help you as students to know what I view as success in mathematics and what I will aim to reward.

1 Introduction

Our overarching goal at university is to *learn how to learn*. We cannot predict where our careers will take us, so we would be foolish to simply pack our brains full with facts and tricks. While we must have a grasp of many such known facts and techniques, we must also be flexible, ready to adjust to our future working environment. That is, we must be ready to efficiently learn things on our own.

Mathematics has an important role in this respect. For in this discipline, we learn how to think logically. We learn how to read mathematics. We learn how to model and solve problems. And we eventually learn how to create mathematics, i.e., how to write proofs. These abilities give us incredible preparation for understanding new technology and digesting technical material in the information age.

2 Start from scratch

To underscore the logical framework in which we work, we start all of our mathematical endeavours from scratch. Mathematicians assume practically nothing; everything is to be verified. (Mathematics is so suspicious of incorrect assumptions that even the natural numbers need to be proved to exist. The in-depth study of assumptions — or "axioms" — and their consistency/dependency is called "Foundations".) For this reason, we rely very heavily on the definitions we give to objects or notations. For example, the statement "p implies q" has its origin in the notion "if p is true, then q is true". But this does not suffice as a definition. (What if p is false?) So we *define* "p implies q" to be a logical statement which is true except when p is true and q is false. We will rely on this definition quite a bit. While the former notion is useful for intuitive reasoning, it is always the definition which we use in real proofs.

A proof is a logical argument which establishes the validity of a certain statement — or "proposition" — usually under given assumptions. A proof is to be read, and in its reading it should instruct the reader. It is usually in paragraph form and consists of a sequence of sentences which convince the reader of the validity of the proposition. These statements break the "leap" from the hypotheses to the conclusion into easily justified steps, each verifiable with little effort by a sufficiently trained reader.

3 Toward mathematical maturity: the three R's

To really learn mathematics, we must all *participate*. Some teachers say: *"Mathematics is not a spectator sport"*. We learn by doing, so exercises are tremendously important. Not only computational examples, but thought-provoking exercises such as proofs and verifications of proofs which involve the student in the very creation of mathematics.

Some students choose mathematics because it doesn't involve reading or writing. To them, mathematics is the manipulation of equations. Well, sorry to say, that is very far from the truth. While much of high school math focuses on computation and manipulation of equations, university mathematics departs dramatically from this. (Refusal to face this fact is a major cause of difficulty for beginning math students.) Yes, like any advanced discipline, communication is at the core of mathematics. To learn, we must be trained to read, and to use what we have learned we must be able to write and to speak clearly.

Reading

Be very patient with yourself when you read mathematics. Read only a few pages in a sitting and read them *very slowly*. Don't skip a single sentence. Be very skeptical: check the author's every step. If any step is unclear to you, try to fill in the missing pieces on scratch paper. If you are still stuck on that sentence, then skip over it and ask a classmate or professor later. Whenever a new definition is introduced, try to construct (or test) familiar examples. After an argument is given, test it on examples using scratch paper. Reading mathematics is like no other reading exercise: it is very time-consuming, but well worth it. It is not unusual for me to proceed at a rate of 10 minutes per sentence in difficult mathematics.

When you read the statement of a theorem in the text, pause before reading the proof. First try to understand what the theorem is saying. Does it really accomplish anything that could be useful? Then try to come up with a proof by yourself. Sometimes you will only get so far, but in other situations, you will be able to predict exactly how the proof will go. This is an accomplishment in which you should take great pride.

The practice of reading proofs helps us in several ways. First off, it reinforces in us the discipline of proceeding slowly and logically through difficult material. Secondly, it involves us in the creation of new mathematics. We are seeing into the minds of great mathematicians, seeing the core of their original ideas. Thirdly, it teaches us how to learn. When we apply mathematics, a theorem that we know or have read may not quite fit. A computer scientist might say that the proof is the "source code" which can be customized and adjusted to cater to unforseen situations.

In terms of the short-term benefit, the ability to read proofs critically enables us to check our own work. At about her second year of university, a math major is expected to have the ability to grade her own work. In this way, a carefully completed and thoroughly checked assignment will meet a very high standard of excellence.

Writing

The practice of writing proofs is an opportunity for us to exercise our creativity in a very pure form. Yes, *creativity!* In an advanced math course, it is possible for twenty students to produce twenty distinct proofs of the same result. Sometimes, a proof is discovered which is entirely original and warrants publicity among the international mathematics community. New ideas such as these can come from anyone and yet can have an impact on the cutting edge of today's research.

At a more concrete level, doing proofs allows us to test whether or not we understand course material. Can we use known facts as building blocks to construct a more complex proof? Do we have the discipline to write statements which follow logically from one to the next?

Let me pause to mention one application. A computer programme is not much different from a proof. If the statements written do not follow from one another logically, there is an error. Sometimes these are found by compilers or by tests performed before commercial release. But some errors survive until the customer discovers them. Even if the statements of a computer programme follow in logical succession, this is no guarantee that they accomplish what they set out to do. Such errors can be very difficult to debug if the writer has no mathematical training.

Communicating

The third aspect of doing mathematics is *communicating mathematics*. By this, I mean verbally explaining a mathematical fact to an audience. While I believe that every one of you should have a chance to present some mathematics to the whole class, most of your experience with this will occur when you are studying with other students. I greatly encourage you to study in groups (not just sharing homework answers, which buys you marks in exchange for lack of understanding). Make arrangements to regularly study the text together. Hold each other accountable to give clear and complete explanations of your thoughts. Don't be afraid to ask one another: *"Why?" "Can you prove it?" "Can you give me an example?"*

Here's an idea: Make an agreement with two friends that you will all read a section — Section 2.3.2, say — of the text and meet the next day. For example, you might sit together at a table and work through the whole section, taking turns explaining to one another subsections of half a page or so.

Ability to communicate is a common measure of understanding. Experts say *"if you can't explain it, then you don't understand it"*.

'Rithmetic

A brief comment about arithmetic and high school algebra. Many university students feel that arithmetic errors are not important and should be forgiven. *"What's the use in practicing arithmetic when we all have calculators and computers?"* In many mathematics subjects, digesting course material often involves small-scale calculations. If a student trying to understand a complex principle of number theory makes an addition error, it will confuse him, and for no good reason. Simply put: we have invested so much energy in these advanced subjects that it would be a shame to risk our understanding on multiplication errors. So be vigilant even with these simple things.

4 Some pointers

At all aspects of mathematics — reading and writing and communicating, even arithemtic — *work fast slowly.* Make sure that the first step is correct before proceeding to the second. Do it right the first time. You may be intimidated by a friend who "finished" in half the time you did, but you will find that he will have to do the whole thing over because of sloppy errors or lack of understanding, and then — if he doesn't change — he'll need to do it a third time! So the fastest way to get things done correctly is to work fast slowly. Have high standards for yourself. Insist on perfection the first time around and it will save you time in the long run.

Our goal in writing a proof is to break a logical leap into small, easily verified steps. Many students have difficulty deciding what constitutes "small".

If $f(x) = x^2 - x + 4$, can we simply say $f(2) = 6$ or do we need to say "Since $2^2 = 4$, $2^2 - 2 = 2$, and so $f(2) = 2^2 - 2 + 4 = 6$". Do we need to prove the distributive law for sets every time we use it?

The rule of thumb here is as follows. *Your proofs should be written so that they are instructive to a weaker student in your class.* Assume that the reader of your proof has a high school education. Assume that she has taken Linear Algebra and some course involving proofs at WPI. So she knows the methods of proof by contrapositive and proof by induction. Assume that she knows the basic definitions of your text (up to the current chapter). Still, in some proofs, there is need to provide definitions of newer or less commonly used terms. You can use theorems from the text, (unless, of course, your task is to prove (part of) that theorem). A good gauge of the level of your treatment is to ask a friend in the class to read your proof. See if they can honestly understand it without any verbal coaching from you. Ask them to critique your proof. Always take and give criticism in a positive way. Without criticism, we can never learn.

How to devise a proof

Briefly, I'll make a few concrete comments about inventing proofs. There is much more to be said, of course (see the references below). In fact, since a proof is a product of individual creativity, in some sense it is preposterous to even suggest that a general method can be given. Nevertheless, for beginners, the following might be a helpful start.

Each proof attempt should have a plan. There should be a method of proof in mind (e.g., proof by contradiction). There is often a key insight which is to be exploited in the proof. When you actually get around to writing out the details, you may or may not find that it works. The point at which you get stuck will give you information. One way to get over such hurdles is to begin looking for counterexamples. Think of an example which satisfies everything you've written but for which the conclusion is false. This may point you to your next step.

In simple situations, there are only three elements to a proof. You must: (i) choose a method; (ii) use the definitions; (iii) use a creative insight. Always prefer a direct proof. So, for example, if you are proving a statement of the form

$$\forall x \in S \ (p(x) \Rightarrow q(x)),$$

you start by saying "Let x be an arbitrary element of S. Assume $p(x)$ holds. Our goal is to show that $q(x)$ holds." Now you've chosen your method. Next, you will need to use the definitions of $p(x)$ and $q(x)$. The facts you obtain in this way from assuming $p(x)$ is true will be your fuel. You must use your ingenuity to deduce (the meaning of) $q(x)$. Feel free to work from both ends.

Know what you know

An essential element of learning how to learn is to *know what you know.* A few years ago, a student came to me saying that he had trouble with a new concept in the course. But, after questioning him, I found that he knew — albeit superficially — the theorem which he thought was confusing him. Instead, I found that he didn't know some definition thirty pages back in the text. Not knowing something — or not fully understanding some principle — is nothing to be embarassed about. We are all in that position. The embarassing thing is this: some people not only don't know something, but they *don't know that they don't know it!* At the time they tried to learn that material, they either worked "fast fastly" or were so afraid of the subject that they falsely convinced themselves that they had mastered it. So three weeks later, when they need to *apply* that knowledge, they suffer and they don't understand why. (Can you imagine how difficult it would be for such a person to prepare for a test?)

If we are ever to be able to learn independently, we must be honest with ourselves about what we know. This is why it is so important to not only read, but to *digest* the material. For example, when you encounter a new definition or new theorem, play with some examples to be sure that you know how it works. *Know what you know and know what you don't know.*

5 Summary

Here, I have briefly discussed my view of what a math student should be learning. I have mentioned some principles which can make the study of mathematics much more enjoyable and less painful. I don't expect that a lot of these principles make complete sense until you've encountered them in practice. So, I simply ask that — in a month or two, you read this essay again. Let it encourage you to go back to basics. The facts that you memorize in this course may not have much application after the final exam. But the discipline of *doing* mathematics — reading, writing, communicating — will benefit anyone who pursues a career in technology or whose career involves logical reasoning.

Inasmuch as a mathematician abides by a strict discipline of logical thinking and thorough absorption of ideas, she is a very creative individual. Creating mathematics — whether discovering patterns, re-proving old theorems, or proving new theorems — is a great achievement of the human mind. It can be fun, suspenseful, and tremendously challenging. Those who put their mind to it should be proud of their effort.

References

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