

Some Categories and Functors

NOTE: This handout is optional material which might allow graduate students to see connections between the different subjects to which they are being introduced. Most of this will not make sense for some time.

A *category* \mathcal{C} consists of a collection of *objects* and sets of *morphisms* (or *arrows*) between objects: to any two objects A, B in \mathcal{C} , our category associates a set $\text{Hom}_{\mathcal{C}}(A, B)$ of morphisms from A to B . These morphisms are assumed to have a *law of composition*

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

taking (f, g) to gf , the “composition” of g with f satisfying the following three axioms:

- (i) If $A \neq C$ or $B \neq D$, then $\text{Hom}_{\mathcal{C}}(A, B)$ and $\text{Hom}_{\mathcal{C}}(C, D)$ are disjoint sets;
- (ii) For every A, B, C, D in \mathcal{C} and every $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$, $\psi \in \text{Hom}_{\mathcal{C}}(B, C)$ and $\sigma \in \text{Hom}_{\mathcal{C}}(C, D)$, we have the associative law

$$\sigma(\psi\varphi) = (\sigma\psi)\varphi ;$$

- (iii) Each object has an identity morphism: for any A in \mathcal{C} , there exists a morphism $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ satisfying (for every object B in our category) $\varphi 1_A = \varphi$ for every $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$ and $1_A \varphi = \varphi$ for every $\varphi \in \text{Hom}_{\mathcal{C}}(B, A)$.

The main goal of this handout is to list some categories that you may have already come across in your studies. (If any one of them is unfamiliar to you, just ignore it.) To avoid paradoxes, we assume all sets are taken from some known pre-specified universe \mathcal{U} .

Name	Objects	Morphisms
Set	all sets in \mathcal{U}	all functions
Top	all topological spaces	continuous functions
Top.	all pointed topological spaces	continuous functions mapping basepoint to basepoint
Grp	all groups	group homomorphisms
Ab	all abelian groups	group homomorphisms
Ring	all rings	ring homomorphisms
k -fdVec	all finite-dim. vector spaces over field k	k -linear transformations
RM	all Riemannian manifolds	e.g., locally isometric maps
Meas	all measurable spaces	measurable morphisms
SimpGph	all undirected simple graphs (reflexive, symmetric adjacency relation)	graph homomorphisms ($u \sim v \Rightarrow f(u) \sim f(v)$)
k -Var	all quasi-projective algebraic varieties over algebraically closed field k	regular maps
Poset	all partially ordered sets	order-preserving maps
G	any group G	one arrow for each $g \in G$ with $g \circ h = gh$
\mathcal{P}	any poset \mathcal{P}	single arrow $x \rightarrow y$ when $x \leq y$
ExactSeq	all short exact sequences	short exact sequence homomorphisms

And here are a few examples of functors.

1. Identity functor: $\mathcal{C} \rightsquigarrow \mathcal{C}$
2. Inclusion functor of subcategories
3. Forgetful functor: for any “concrete” category \mathcal{C} , we map each object to its underlying set $\mathcal{C} \rightsquigarrow \mathbf{Set}$
4. Abelianizing functor: $\mathbf{Grp} \rightsquigarrow \mathbf{Ab}$ via $G \rightsquigarrow G/G'$
5. Group Ring: $\mathbf{Grp} \rightsquigarrow \mathbf{Ring}$
6. Forget multiplication: $\mathbf{Ring} \rightsquigarrow \mathbf{Ab}$
7. Fundamental group: $\mathbf{Top} \rightsquigarrow \mathbf{Grp}$
8. k^{th} homology group: $\mathbf{Top} \rightsquigarrow \mathbf{Ab}$