MA535 ALGEBRA
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The Subgroup Lattice

Posets

A *partially ordered set* is an ordered pair $(\mathcal{P}, \preceq)$ where $\mathcal{P}$ is a set and $\preceq$ is a binary relation on $\mathcal{P}$ which is

- *reflexive*: for all $a \in \mathcal{P}$, $a \preceq a$
- *antisymmetric*: for all $a, b \in \mathcal{P}$, if $a \preceq b$ and $b \preceq a$, then $a = b$, and
- *transitive*: for all $a, b, c \in \mathcal{P}$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Instead of writing out *"partially ordered sets"*, we usually just call these *"posets"*. Naturally, the notation $a \prec b$ means $a \preceq b$ but $a \neq b$.

If G is any set and $\mathcal{H}$ is any collection of subsets of G, these can be partially ordered by inclusion. I.e., In this case $(\mathcal{H}, \subseteq)$ is a partial order relation.

We are interested in the poset of all subgroups of a group G. The example handed out in class depicts this poset for the symmetry group of a hexagon, D_{12}.

Lattices

A lattice[1] is a special kind of poset. A *lattice* is a partially ordered set in which every two elements have a well-defined *meet* and a well-defined *join*. Let $(\mathcal{P}, \preceq)$ be a poset. For $a, b, c \in \mathcal{P}$, we say c is a *meet* for a and b (and we write $c = a \wedge b$) if

- $c \preceq a$ and $c \preceq b$, and
- for all $x \in \mathcal{P}$, if $x \preceq a$ and $x \preceq b$, then $x \preceq c$

(So we view c as the "greatest lower bound" of a and b.) Familiar examples include set intersections and greatest common divisors. It is an easy exercise to show that, if a meet for a and b exists, then it is unique; so $a \wedge b$ is well-defined.

On the other hand, for $a, b, c \in \mathcal{P}$, we say c is a *join* for a and b (and we write $c = a \vee b$) if

- $a \preceq c$ and $b \preceq c$, and
- for all $x \in \mathcal{P}$, if $a \preceq x$ and $b \preceq x$, then $c \preceq x$

[1] Beware! The term "lattice" is used in mathematics for two completely different concepts.

This is viewed as a "least upper bound" and we have familiar examples of union and l.c.m.; if $a \vee b$ exists, then it is uniquely determined.

The Subgroup Lattice

The subgroups of a group G form a lattice because, for any two subgroups H and K of G, $H \cap K$ is a subgroup (and therefore gives us the meet) and $\langle H, K \rangle$, the smallest subgroup containing both H and K acts as a join (though not as elegantly defined as we might hope it to be).

The Hasse Diagram

This usually only makes sense when G is finite, but we depict a poset $(\mathcal{P}, \preceq)$ on a piece of paper by writing down the elements of $\mathcal{P}$ from "smallest" to "biggest"; i.e., if $a \prec b$, we try to draw b higher up on the page than a. We use "edges" (line segments or arcs) to indicate the relation $\prec$. Rather than clutter up the page with all possible ordered pairs in $\preceq$, we only draw the *covering relation*: we say b *covers* a if $a \prec b$ and, for all $c \in \mathcal{P}$, if $a \preceq c \preceq b$, then either $c = a$ or $c = b$. (Obviously, this makes perfect sense for the integers under their natural total order, but the cover relation is empty for the rationals or reals under $<$.)

Our *Hasse diagram* draws an edge from a to b whenever b covers a. An example is attached.