MA535 ALGEBRA
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## Some Categories and Functors

NOTE: This handout is optional material which might allow graduate students to see connections between the different subjects to which they are being introduced. Most of this will not make sense for some time.

A *category* $\mathcal{C}$ consists of a collection of *objects* and sets of *morphisms* (or *arrows*) between objects: to any two objects $A, B$ in $\mathcal{C}$, our category associates a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms from $A$ to $B$. These morphisms are assumed to have a *law of composition*

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

taking $(f, g)$ to $gf$, the "composition" of $g$ with $f$ satisfying the following three axioms:

(i) If $A \neq C$ or $B \neq D$, then $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(C, D)$ are disjoint sets;

(ii) For every $A, B, C, D$ in $\mathcal{C}$ and every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $\sigma \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, we have the associative law

$$\sigma(\psi\varphi) = (\sigma\psi)\varphi \ ;$$

(iii) Each object has an identity morphism: for any $A$ in $\mathcal{C}$, there exists a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ satisfying (for every object $B$ in our category) $\varphi 1_A = \varphi$ for every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $1_A\varphi = \varphi$ for every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.

The main goal of this handout is to list some categories that you may have already come across in your studies. (If any one of them is unfamiliar to you, just ignore it.) To avoid paradoxes, we assume all sets are taken from some known pre-specified universe $\mathcal{U}$.

| Name | Objects | Morphisms |
|---|---|---|
| Set | all sets in $\mathcal{U}$ | all functions |
| Grp | all groups | group homomorphisms |
| Ab | all abelian groups | group homomorphisms |
| Ring | all rings | ring homomorphisms |
| $k$-fdVec | all finite-dim. vector spaces over field $k$ | $k$-linear transformations |
| Top | all topological spaces | continuous functions |
| SimpGph | all undirected simple graphs (reflexive, symmetric adjacency relation) | graph homomorphisms $(u \sim v \Rightarrow f(u) \sim f(v))$ |
| $k$-Var | all quasi-projective algebraic varieties over algebraically closed field $k$ | regular maps |
| RM | all Riemannian manifolds | e.g., locally isometric maps |
| Meas | all measurable spaces | measurable morphisms |
| Poset | all partially ordered sets | order-preserving maps |
| $G$ | any group $G$ | one arrow for each $g \in G$ with $g \circ h = gh$ |
| $\mathcal{P}$ | any poset $\mathcal{P}$ | single arrow $x \to y$ when $x \leq y$ |
| ExactSeq | all short exact sequences | short exact sequence homomorphisms |

And here are a few examples of functors.

1. Identity functor: $\mathcal{C} \rightsquigarrow \mathcal{C}$
2. Inclusion functor of subcategories
3. Forgetful functor: for any "concrete" category $\mathcal{C}$, we map each object to its underlying set $\mathcal{C} \rightsquigarrow \mathsf{Set}$
4. Abelianizing functor: $\mathsf{Grp} \rightsquigarrow \mathsf{Ab}$ via $G \rightsquigarrow G/G'$
5. Group Ring: $\mathsf{Grp} \rightsquigarrow \mathsf{Ring}$
6. Forget multiplication: $\mathsf{Ring} \rightsquigarrow \mathsf{Ab}$
7. Fundamental group: $\mathsf{Top} \rightsquigarrow \mathsf{Grp}$
8. $k^{\text{th}}$ homology group: $\mathsf{Top} \rightsquigarrow \mathsf{Ab}$