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Partial Fractions Expansion of Rational Functions

An Application of the Fundamental Theorem of Algebra

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The Fundamental Theorem of Algebra is important throughout mathematics. Here are two equivalent statements of this theorem.

Theorem 1 (Fundamental Theorem of Algebra, Building Block)**.** *Every non-constant polynomial with complex coefficients has a complex root.*

Theorem 2 (Fundamental Theorem of Algebra, Factorization)**.** *A non-zero polynomial of degree n with complex coefficients has exactly n complex roots, counting multiplicities.*

Let $\mathbb{C}[z]$ denote the ring of polynomials with complex coefficients. Each element $f(z) \in \mathbb{C}[z]$ has the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

for some complex numbers $a_0, a_1, \ldots, a_n$. Provided $a_n \neq 0$, we say $f(z)$ has degree n. We say $f(z)$ is a *monic* polynomial if $a_n = 1$ (the leading coefficient is equal to one).

The first version of the FTA states that, if $f(z) \in \mathbb{C}[z]$ has degree $n \geq 1$, then there is some complex number $c \in \mathbb{C}$ such that

$$f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 = 0.$$

By the Factor Theorem, this tells us that the polynomial $z - c$ divides $f(z)$. So there is some monic polynomial $g(z)$ such that

$$f(z) = a_n (x - c) g(z).$$

Since $\mathbb{C}$ is an integral domain (in fact, it's a field), we know that $g(z)$ has degree $n-1$. Applying the Principle of Mathematical Induction, we may assume that $g(z)$ has exactly $n-1$ complex roots. So the "building block" version of the FTA gives us the factorization version of the FTA:

> For any polynomial $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients, with $n \geq 0$ and $a_n \neq 0$, there exist n complex numbers
>
> $$r_1, r_2, \ldots, r_n$$
>
> not necessarily distinct, such that $f(z)$ factors into irreducibles as
>
> $$f(z) = a_n (z - r_1)(z - r_2) \cdots (z - r_n).$$

The Fundamental Theorem of Algebra quickly implies (and it is essentially equivalent to) the following useful theorem about polynomials with real coefficients. The ring $\mathbb{R}[x]$ consists of all polynomials with real coefficients, that is, all

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \geq 0$ and all a_i are real numbers.

Quadratic polynomials $f(x) = ax^2 + bx + c$ fall into two classes: those with real roots[1] (which happens when $b^2 - 4ac \geq 0$) and those with no real roots (which happens when $b^2 - 4ac < 0$). These latter type are called "irreducible quadratics".

Theorem 3 (Fundamental Theorem of Algebra, Real Coefficients)**.** *Every nonzero polynomial with real coefficients factors uniquely into linear polynomials and irreducible quadratics.*

Let's unwrap this and spell out exactly what it tells us.

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 0$ and $a_n \neq 0$, then there are unique (up to ordering) monic polynomials $\ell_1(x) = x - r_1$, $\ell_2(x) = x - r_2$, $\ldots \ell_s(x) = x - r_s$, and $q_1(x) = x^2 + b_1 x + c_1$, $q_2(x) = x^2 + b_2 x + c_2$, $\ldots q_t(x) = x^2 + b_t x + c_t$ with each $b_j^2 - 4c_j < 0$ (and, comparing degrees, $s + 2t = n$) such that

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_s)(x^2 + b_1 x + c_1)(x^2 + b_2 x + c_2)\cdots(x^2 + b_t x + c_t).$$

We then see that this polynomial has exactly s real roots (namely, $r_1, r_2, \ldots, r_s$) and $2t$ complex roots that are not real:

$$\frac{-b_1}{2} + \frac{i}{2}\sqrt{4c_1 - b_1^2},\ \frac{-b_1}{2} - \frac{i}{2}\sqrt{4c_1 - b_1^2},\ \ldots,\ \frac{-b_t}{2} + \frac{i}{2}\sqrt{4c_t - b_t^2},\ \frac{-b_t}{2} - \frac{i}{2}\sqrt{4c_t - b_t^2}.$$

To prove this result using the FTA, we observe that complex conjugation (sending $z = a + bi$ to $\bar{z} = a - bi$) is a ring isomorphism (one-to-one and onto) from the field of complex numbers to itself:

$$\overline{z + w} = \bar{z} + \bar{w}, \qquad \overline{z \cdot w} = \bar{z} \cdot \bar{w},$$

with $\bar{z} = z$ if and only if z is a real number.

Suppose that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with real coefficients and suppose that the complex number z is a root of f. Then

$$0 = f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

and, applying complex conjugation to both sides,

$$0 = \bar{0} = \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \cdots + a_1 \bar{z} + a_0$$

showing that $\bar{z}$ is also a root of $f(x)$. So the non-real roots of $f(x)$ come in conjugate pairs.

It is important to remember that, for any complex number z, both $z + \bar{z}$ and $z \cdot \bar{z}$ are real numbers. (Please check this.) So the polynomial $(x - z)(x - \bar{z})$ has the form $x^2 + bx + c$ where b and c are real numbers.

What does this achieve for us? We are assuming that $f(x)$ has real coefficients. We apply the Fundamental Theorem of Algebra to express $f(x) = a_n x^n + \cdots + a_1 x + a_0$ as

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

[1]Note that the quadratic $f(x) = ax^2 + bx + c$ has a double root (or repeated root) precisely when $b^2 - 4ac = 0$.

where $r_1, \ldots, r_n$ are the n complex roots of f. We can re-order these so that the real roots come first: say $r_1, \ldots, r_s$ are real numbers and all the rest are non-real complex numbers. We just learned that these come in conjugate pairs. So $n - s$ is even and we can re-order terms so that r_{s+2} is the conjugate of r_{s+1} and r_{s+4} is the conjugate of r_{s+3} and so on. But then each pair of linear factors $(x - r_{s+j-1})(x - r_{s+j})$ where j is even expands into an irreducible quadratic polynomial with real coefficients: since $r_{s+j-1} = \bar{r}_{s+j}$, we have

$$(x - r_{s+j-1})(x - r_{s+j}) = x^2 - (r_{s+j} + \bar{r}_{s+j})x + r_{s+j}\bar{r}_{s+j}.$$

This is the origin of the irreducible quadratic factors in the "real" version of the FTA.

1 Partial Fractions Expansion

The ring of rational functions contains all ratios of two polynomials in x

$$\left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x), g(x) \in \mathbb{R}[x],\ g(x) \text{ not the zero polynomial} \right\}$$

is a commutative ring with operations

$$\frac{f(x)}{g(x)} + \frac{h(x)}{k(x)} = \frac{f(x)k(x) + h(x)g(x)}{g(x)k(x)} \quad \text{and} \quad \frac{f(x)}{g(x)} \cdot \frac{h(x)}{k(x)} = \frac{f(x)h(x)}{g(x)k(x)}.$$

But this happens to be a field! Every non-zero element has an inverse. Let's check. Suppose r is a rational function which is not the zero element. Then r takes the form $\frac{f(x)}{g(x)}$ for some polynomials $f(x)$ and $g(x)$ and r being non-zero means that $f(x)$ is not the zero polynomial. So the ratio $\frac{g(x)}{f(x)}$ is also a rational function and it is easy to check that this is the inverse of $\frac{f(x)}{g(x)}$.

Since 1 is a non-zero polynomial, we see that every polynomial is a rational function: write $f(x)$ as $\frac{f(x)}{1}$. (In fact, the identity element 1 for this field is technically written $\frac{1}{1}$.) By long division (the Division Algorithm for polynomials), we can express any rational function $f(x)/g(x)$ in the form $q(x) + r(x)/g(x)$ where the degree of r is less than the degree of g. For there is a unique quotient $q(x)$ and a unique remainder $r(x)$ such that

- $f(x) = g(x)q(x) + r(x)$
- either $r(x)$ is the zero polynomial or $\deg r < \deg g$

What we are after is a way to reverse the common denominator approach to adding fractions. We can all easily compute

$$\frac{3}{7} + \frac{2}{5} = \frac{29}{35}$$

but how do we find small integers a and b such that

$$\frac{17}{33} = \frac{a}{3} + \frac{b}{11} \ ?$$

Do we even know that such a and b exist? We can multiply both sides by 33 to get

$$17 = 11a + 3b.$$

Since 11 and 3 are relatively prime, we can find integers x and y such that

$$11x + 3y = 1.$$

For example $x = -1, y = 4$ works. We can then multiply both sides of this equation by 17 to find

$$17 = 11(-17) + 3(68)$$

but the expansion

$$\frac{17}{33} = -\frac{17}{3} + \frac{68}{11}$$

doesn't seem like much of a simplification! We'd rather have $\frac{17}{33} = \frac{1}{3} + \frac{2}{11}$. [I'll leave this mystery to you to resolve.]

We want to do the same thing for polynomials: "undo" the common denominator calculation. For instance, while we all can compute

$$\frac{3}{x+5} - \frac{2}{x-5} = \frac{x-25}{x^2-25}$$

we must solve a system of linear equations to discover that

$$\frac{12}{x^2-x-2} = \frac{4}{x-2} - \frac{4}{x+1}.$$

Example: Find real numbers A and B such that

$$\frac{4x+1}{x^2-3x-10} = \frac{A}{x+2} + \frac{B}{x-5}.$$

Solution: Multiply both sides by $x^2 - 3x - 10 = (x+2)(x-5)$ to find

$$4x + 1 = (A + B)x + (2B - 5A).$$

In order for these polynomials to be equal, all corresponding coefficients must be equal. So we obtain the linear system

$$A + B = 4, \qquad 2B - 5A = 1$$

which we quickly solve to find $A = 1$, $B = 3$. □

The fact that this can always be achieved follows from the following lemmas. Comparing to our handling of mixed numbers, the first lemma is analogous to saying that, if m and n are integers with $0 \le n < 7$ which satisfy

$$5\tfrac{2}{7} = m + \frac{n}{7},$$

then $m = 5$ and $n = 2$.

Lemma 1. *If $f_1, g_1, h_1, f_2, g_2, h_2$ are all polynomials with $\deg g_1 < \deg h_1$ and $\deg g_2 < \deg h_2$ and the equation*

$$f_1(x) + \frac{g_1(x)}{h_1(x)} = f_2(x) + \frac{g_2(x)}{h_2(x)}$$

holds, then $f_1(x) = f_2(x)$ and $h_2(x)g_1(x) = h_1(x)g_2(x)$.

Proof: Omitted. □

Lemma 2. *If $f(x)$ and $g(x)$ are polynomials with greatest common divisor $d(x)$, then there exist polynomials $s(x)$ and $t(x)$ satisfying the equation*

$$f(x) \cdot s(x) + g(x) \cdot t(x) = d(x).$$

In particular, polynomials f and g have no common factor (i.e., the two polynomials are relatively prime) if and only if there exist polynomials $s(x)$ and $t(x)$ satisfying the equation $f(x) \cdot s(x) + g(x) \cdot t(x) = 1$.

Proof: This follows from the Extended Euclidean Algorithm much in the same way as the analogous theorem for integers. □

Lemma 3. *Suppose we are given a rational function $f(x)/g(x)$ where $\deg f < \deg g$. If $g(x)$ factors as $g(x) = p(x) \cdot q(x)$ where polynomials $p(x)$ and $q(x)$ are relatively prime, then there exist polynomials $s(x)$ and $t(x)$ such that*

- $\dfrac{f(x)}{g(x)} = \dfrac{s(x)}{p(x)} + \dfrac{t(x)}{q(x)}$
- $\deg s < \deg p$ *and* $\deg t < \deg q$.

Proof: Since $p(x)$ and $q(x)$ have gcd equal to 1, there are polynomials $S(x)$ and $T(x)$ such that $S(x)q(x) + T(x)p(x) = 1$. Multiplying both sides by the rational function $f(x)/g(x)$, we get

$$\frac{S(x)f(x)}{p(x)} + \frac{T(x)f(x)}{q(x)} = \frac{f(x)}{g(x)} \tag{1}$$

since $g(x) = p(x) \cdot q(x)$. Now it may be that one of the two rational functions on the left has a numerator with higher degree than its denominator. If $\deg S + \deg f \geq \deg p$, then write

$$S(x)f(x) = p(x)u(x) + r(x)$$

where $\deg r < \deg p$; otherwise, take $u(x) = 0$ and $r(x) = S(x)f(x)$. Likewise, write

$$T(x)f(x) = q(x)w(x) + v(x)$$

for some unique quotient $w(x)$ (possibly zero) and some remainder $v(x)$ with $\deg v < \deg q$. Then Equation (1) can be written

$$u(x) + \frac{r(x)}{p(x)} + w(x) + \frac{v(x)}{q(x)} = \frac{f(x)}{g(x)}$$

and by Lemma 1, we have $u(x)+w(x)$ is the zero polynomial since $\deg f < \deg g$. Eliminating these two terms gives us the desired decomposition. $\square$

(You might now wish to go back to the equation $\frac{17}{33} = -\frac{17}{3} + \frac{68}{11}$ and see how this idea allows us to simplify it to $\frac{17}{33} = \frac{1}{3} + \frac{2}{11}$.

Lemma 4. *Consider rational functions where the denominator is expressible as some irreducible polynomial in $\mathbb{R}[x]$ to some power.*

(a) If $\deg f(x) < k$ and $c \in \mathbb{R}$, then there exist unique real numbers $a_k, a_{k-1}, \ldots, a_1$ such that

$$\frac{f(x)}{(x-c)^k} = \frac{a_k}{(x-c)^k} + \frac{a_{k-1}}{(x-c)^{k-1}} + \cdots + \frac{a_2}{(x-c)^2} + \frac{a_k}{x-c}.$$

(b) If $\deg f(x) < 2k$ and $b, c \in \mathbb{R}$ with $b^2 - 4c < 0$, then there exist unique real numbers $A_k, A_{k-1}, \ldots, A_1$ and $B_k, B_{k-1}, \ldots, B_1$ such that

$$\frac{f(x)}{(x^2+bx+c)^k} = \frac{A_k x + B_k}{(x^2+bx+c)^k} + \frac{A_{k-1}x + B_{k-1}}{(x^2+bx+c)^{k-1}} + \cdots + \frac{A_1 x + B_1}{x^2+bx+c}.$$

Proof: Repeatedly apply the Division Algorithm (long division) for polynomials: in case (a),

$$f(x) = (x-c)f_k(x) + a_k$$

so that

$$\frac{f(x)}{(x-c)^k} = \frac{a_k}{(x-c)^k} + \frac{f_k(x)}{(x-c)^{k-1}}$$

and write $f_k(x) = (x-c)f_{k-1}(x) + a_{k-1}$ so that $\frac{f_k(x)}{(x-c)^{k-1}} = \frac{a_{k-1}}{(x-c)^{k-1}} + \frac{f_{k-1}(x)}{(x-c)^{k-2}}$ and so on. $\square$

Theorem 4. *Let $f(x)/g(x)$ be any rational function which is a quotient of polynomials with real coefficients. Factor $g(x)$ as*

$$g(x) = b_n(x-r_1)^{k_1}(x-r_2)^{k_2}\cdots(x-r_s)^{k_s}(x^2+b_1x+c_1)^{\ell_1}(x^2+b_2x+c_2)^{\ell_2}\cdots(x^2+b_tx+c_t)^{\ell_t}.$$

Then there exists a unique polynomial $H(x)$ and unique real numbers

$$A_{k_1,1}, A_{k_1-1,1}, \ldots, A_{1,1}, \quad A_{k_2,2}, \ldots, A_{1,2}, \quad \ldots, A_{k_s,s}, \ldots, A_{1,s},$$
$$B_{\ell_1,1}, B_{\ell_1-1,1}, \ldots, B_{1,1}, \quad B_{\ell_2,2}, \ldots, B_{1,2}, \quad \ldots, B_{\ell_t,t}, \ldots, B_{1,t},$$
$$C_{\ell_1,1}, C_{\ell_1-1,1}, \ldots, C_{1,1}, \quad C_{\ell_2,2}, \ldots, C_{1,2}, \quad \ldots, C_{\ell_t,t}, \ldots, C_{1,t},$$

such that

$$\frac{f(x)}{g(x)} = H(x) + \frac{A_{k_1,1}}{(x-r_1)^{k_1}} + \frac{A_{k_1-1,1}}{(x-r_1)^{k_1-1}} + \cdots + \frac{A_{1,1}}{x-r_1} + \frac{A_{k_2,2}}{(x-r_2)^{k_2}} + \cdots + \frac{A_{1,2}}{x-r_2} + \cdots$$
$$+ \frac{A_{k_s,s}}{(x-r_s)^{k_s}} + \cdots + \frac{A_{1,s}}{x-r_s} + \frac{B_{\ell_1,1}x + C_{\ell_1,1}}{(x^2+b_1x+c_1)^{\ell_1}} + \frac{B_{\ell_1-1,1}x + C_{\ell_1-1,1}}{(x^2+b_1x+c_1)^{\ell_1-1}} + \cdots + \frac{B_{1,1}x + C_{1,1}}{x^2+b_1x+c_1}$$
$$+ \frac{B_{\ell_2,2}x + C_{\ell_2,2}}{(x^2+b_2x+c_2)^{\ell_2}} + \cdots + \frac{B_{1,2}x + C_{1,2}}{x^2+b_2x+c_2} + \cdots + \frac{B_{\ell_t,t}x + C_{\ell_t,t}}{(x^2+b_tx+c_t)^{\ell_t}} + \cdots + \frac{B_{1,t}x + C_{1,t}}{(x^2+b_tx+c_t}.$$

Proof: This horrendous expression is obtained by applying the previous lemmas. First, if the degree of $f(x)$ is larger or equal to the degree of $g(x)$, then we apply the Division Algorithm to get the quotient $H(x)$ and a remainder of degree less than $\deg g$. So, for the remainder of this proof, we assume the numerator has smaller degree than the denominator.

Whenever $g(x)$ factors, we can either break it up into relatively prime polynomials and apply Lemma 3 to write the fraction $f(x)/g(x)$ as a sum of two fractions (rational functions) whose denominators have smaller degrees OR we have only repeated factors: $g(x) = p(x)^k$ for some polynomial $p(x)$ and some integer $k \geq 1$. We only get stuck here when $p(x)$ is irreducible in the ring $\mathbb{R}[x]$. By the FTA, we know that $p(x)$ is either a linear polynomial $x - r$ or an irreducible quadratic polynomial $x^2 + bx + c$ with $b^2 - 4c < 0$. In this case, we apply Lemma 4 to decompose $f(x)/(p(x))^k$ into a sum of terms $A/(x-r)^j$ (if $p(x) = x - r$) or $(Bx+C)/(x^2+bx+c)^j$ (if $p(x)$ is quadratic. This eventually produces the partial fractions expansion promised by the theorem. $\square$

Of course, the general expression is horrendous, so we should see examples to understand all the notation.

Suppose $f(x) = x^6 - 3x^4 + 8x^3 + x$ and $g(x) = x(x-5)^3(x^2+x+1)^2$. Then there exist unique real numbers

$$A_{1,1}, A_{3,2}, A_{2,2}, A_{1,2}, \quad B_{2,1}, C_{2,1}, B_{1,1}, C_{1,1}$$

such that

$$\frac{x^6 - 3x^4 + 8x^3 + x}{x(x-5)^3(x^2+x+1)^2} = \frac{A_{1,1}}{x} + \frac{A_{3,2}}{(x-5)^3} + \frac{A_{2,2}}{(x-5)^2} + \frac{A_{1,2}}{x-5} + \frac{B_{2,1}x + C_{2,1}}{(x^2+x+1)^2} + \frac{B_{1,1}x + C_{1,1}}{x^2+x+1}.$$

In order to find these numbers, we multiply through both sides by $g(x)$ to remove all denominators and then collect like terms. Matching coefficients of the polynomial on the left to coefficients of the polynomial on the right, term by term, we obtain a system of linear equations which we then solve for the $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$.

The ten exercises for Homework 5 are on the next page.

Exercises

1. RSA encryption is based on algebra in the ring $\mathbb{Z}_n$ where $n = pq$ is a product of two large distinct primes p and q. Security is compromised if p and/or q is leaked. So our hope for security rests on the assumption that factoring the integer n is very hard. The proof that decryption works depends on the integer $\phi(n)$ where ϕ is Euler's totient function.
   (a) Show how to compute $\phi(n)$ using n, p and q.
   (b) Using the quadratic formula, show how to compute p and q if given n and $\phi(n)$.

2. Factor $f(x) = x^4 + 81$ into irreducibles over $\mathbb{R}$.

3. If z is the complex number given by $z = Re^{i\theta}$ where $R > 0$ and θ are real numbers, what is the equivalent expression for the complex number z^n where n is a positive integer? What about $\sqrt{z}$?

4. Write $\frac{1}{x-2} - \frac{3}{x-3} + \frac{2}{x-4}$ as a rational function $f(x)/g(x)$.

5. Compute the partial fractions expansion of the rational function $(x^2-7)/(x^2-7x+10)$ and note that, in this case the polynomial $H(x)$ is not zero.

6. Compute the partial fractions expansion of the rational function $(x^2 + 2x)/(x^3 - 1)$.

7. Compute the partial fractions expansion of $(x^4 - x + 1)/[x^2(x^2 + 1)^2]$. Be careful!

8. Derive the quadratic formula by solving each of the following quadratic equations for x.

   (a) $ax^2 = n$
   (b) $x^2 + 2kx + k^2 = n$
   (c) $a(x + k)^2 = n$
   (d) $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$

9. Write down the first five rows of Pascal's Triangle and, beside this, write down the expansion of the polynomials $(x + y)^1$, $(x + y)^2$, $(x + y)^3$ and $(x + y)^4$.

10. Let $f(x) = x^n$ and let a be a real number. Show that

$$f(x+a) - f(a) = x \cdot \left[nx^{n-1} + \binom{n}{2} x^{n-2} + \cdots + \binom{n}{n-1} \right].$$

    You may use the Binomial Theorem in your derivation without proving it.