Algebra for Educators
E Term 2017
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MME529 Homework 4

Due Date: Tuesday June 20th, 2017

RECALL: BASIC RULES FOR ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only <u>one side</u> of each sheet of paper;

III) Write at a level that will be clear to a classmate; there is no need to justify arithmetic or review definitions covered in class, but explanations should be given for anything that is not obvious. Write in full sentences.

IV) In general, late assignments will not be accepted for credit.

1. # 2 on p300 in the Gallian text
2. # 12 on p301
3. # 13 on p301
4. # 13 on p317
5. Let $\tau = \frac{1}{2}(1+\sqrt{5})$ denote the "Golden ratio". Factor $f(x) = x^2 - x - 1$ in the ring $\mathbb{R}[x]$ and compare this to the factorization of $f(x)$ in the ring $\mathbb{Z}[x]$.
6. Factor $h(x) = x^3 - x^2 + 2$ in the ring $\mathbb{R}[x]$ and compare this to the factorization of $h(x)$ in the ring $\mathbb{Z}[x]$.
7. The complex *fifth roots of unity* are the five complex numbers z satisfying $z^5 = 1$. (The Fundamental Theorem of Algebra ensures us that there are exactly five solutions in $\mathbb{C}$.) These numbers,

$$e^{2\pi i/5}, \quad e^{4\pi i/5}, \quad e^{6\pi i/5}, \quad e^{8\pi i/5}, \quad e^{10\pi i/5} = 1,$$

form a regular pentagon inscribed in the unit circle in the complex plane. All except the last one of these are called *primitive complex fifth roots of unity* because they do not satisfy any equation $z^k = 1$ for k smaller than five. So while the polynomial $g(z) = z^5 - 1$ factors over $\mathbb{C}$ as

$$g(z) = (z-1)(z-e^{2\pi i/5})(z-e^{4\pi i/5})(z-e^{6\pi i/5})(z-e^{8\pi i/5}),$$

its factorization over the real numbers is

$$g(z) = (z-1)(z^2 + \tau z + 1)(z^2 + (1-\tau)z + 1)$$

and its factorization over the integers is simply $g(z) = (z-1)(z^4 + z^3 + z^2 + z + 1)$.

(For another example, there are only two primitive complex sixth roots of unity since the solutions to the equation $z^6 - 1 = 0$ include ± 1 (blue) and two primitive complex cube roots of unity (green), as the regular hexagon at the end of this handout illustrates.)

Keeping all of this (and Euler's totient function!) in mind, consider the polynomial

$$f(x) = x^{10} - 1.$$

(a) Factor $f(x)$ into irreducibles over the complex numbers $\mathbb{C}$

(b) Factor $f(x)$ into irreducibles over the real numbers $\mathbb{R}$ (i.e., factor in the ring $\mathbb{R}[x]$)

(c) Factor $f(x)$ into irreducibles over the integers $\mathbb{Z}$ (i.e., factor in the ring $\mathbb{Z}[x]$)

8. Recall that a *field* is an integral domain in which every nonzero element is a unit (has an inverse). It is an amazing fact that for every prime power $q = p^r$ (p prime, r any positive integer), there is exactly one field of size q, up to isomorphism.

   Construct a multiplication table for a field of order four. Let $\mathbb{F}_4 = \{0, 1, a, b\}$ with addition table given here and fill in the (unique!) multiplication table (copying these two onto the sheet you submit).

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0				
1				
a				
b				

Our last two exercises (below) deal with a famous puzzle in statistics.

Simpson's Paradox in statistics is a phenomenon in which our intuitive temptation to "average averages" leads us to doubt the veracity of data. A standard example involves comparing two softball players (or baseball players who perform differently in domed versus outdoor stadiums). Suppose *Seta* and *Tian* both record their batting averages (hits divided by at-bats) for home games and away games. It is quite possible to arrive at the following batting average data:

Player	Seta	Tian
Home Games	.392	.375
Away Games	.257	.250
All Games	.319	.350

Many will object that Tian can't possibly be a better hitter than Seta when she does worse both at home and playing in away games. This is called *Simpson's Paradox*. It's resolution comes down to simple algebra.

Let's define eight variables:

Statistic	Seta	Tian
Hits: Home Games	h_S	h_T
At-bats: Home Games	b_S	b_T
Hits: Away Games	h'_S	h'_T
At-bats: Away Games	b'_S	b'_T

Then, for example, Seta's overall batting average is given by her total number hits divided by her total number of at-bats $(h_S + h'_S)/(b_S + b'_S)$ and so on. For example, the following table of results

Player	Seta	Tian
Home Games	.500	.333
Away Games	.999	.970
All Games	.625	.950

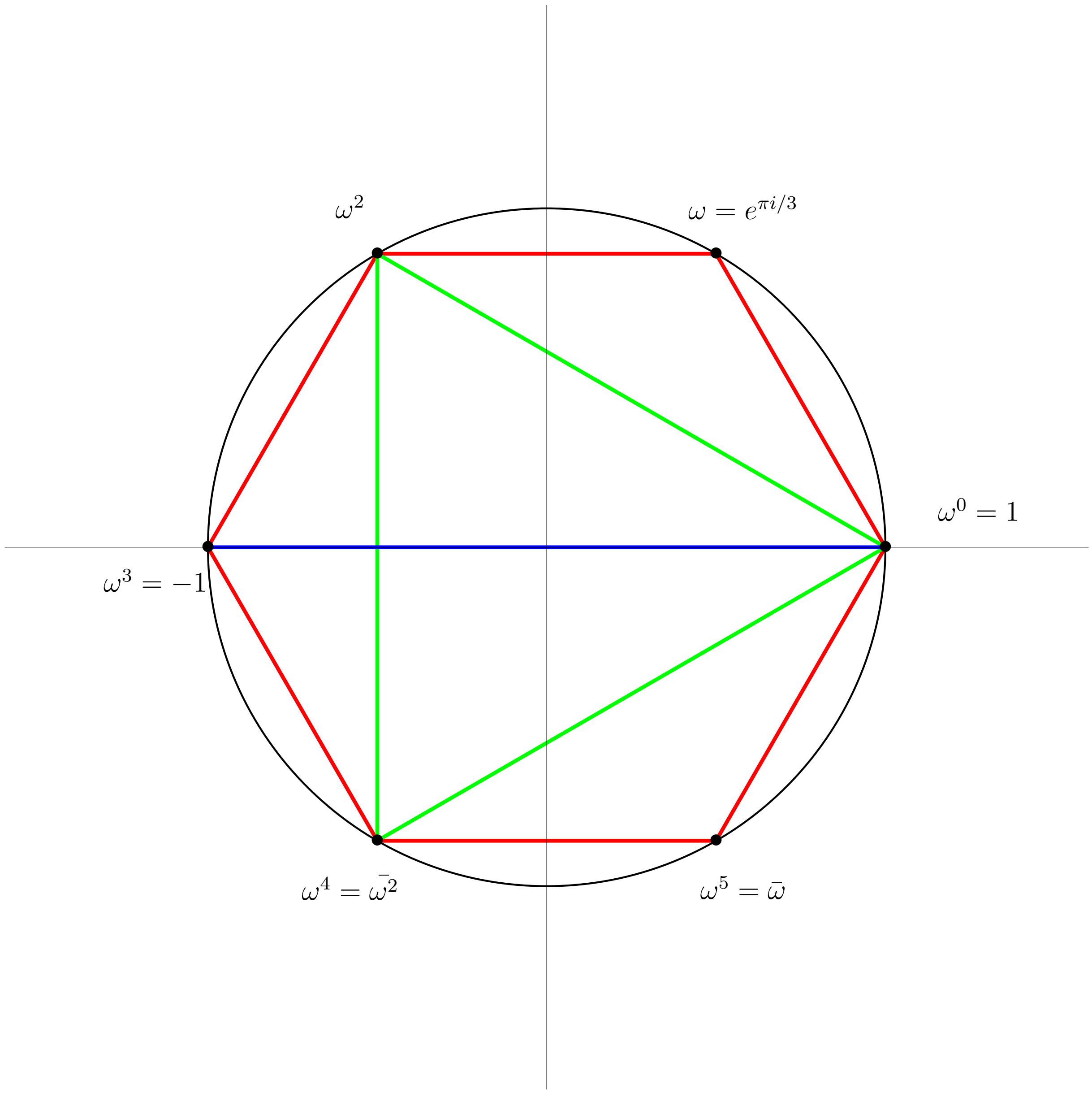
arises from (among other solutions)

$$h_S = 6,\quad h'_S = 4,\quad b_S = 12,\quad b'_S = 4,\qquad\qquad h_T = 1,\quad h'_T = 94,\quad b_T = 3,\quad b'_T = 97.$$

9. Your first task is to find values for these eight variables that lead to the following statistics:

Player	Seta	Tian
Home Games	.800	.750
Away Games	.500	.400
All Games	.650	.680

10. Your second exercise related to Simpson's Paradox is to create an example of your own in which all variable values are single-digit numbers. Your inequalities should reflect that Tian outperforms Seta both home and away and yet has a lower batting average overall.


ω^2
$\omega = e^{\pi i/3}$
$\omega^0 = 1$
$\omega^3 = -1$
$\omega^4 = \bar{\omega^2}$
$\omega^5 = \bar{\omega}$